# BIMONOIDS FOR HYPERPLANE ARRANGEMENTS

The goal of this monograph is to develop Hopf theory in a new setting which features centrally a real hyperplane arrangement. The new theory is parallel to the classical theory of connected Hopf algebras, and relates to it when specialized to the braid arrangement. Joyal's theory of combinatorial species, ideas from Tits' theory of buildings, and Rota's work on incidence algebras inspire and find common ground in this theory.

The authors introduce notions of monoid, comonoid, bimonoid, and Lie monoid relative to a fixed hyperplane arrangement. Faces, flats, and lunes of the arrangement provide the building blocks for these concepts. They also construct universal bimonoids by using generalizations of the classical notions of shuffle and quasishuffle, and establish the Borel–Hopf, Poincaré–Birkhoff–Witt, and Cartier–Milnor–Moore theorems in this new setting. A key role is played by noncommutative zeta and Möbius functions which generalize the classical exponential and logarithm, and by the representation theory of the Tits algebra.

This monograph opens a vast new area of research. It will be of interest to students and researchers working in the areas of hyperplane arrangements, semigroup theory, Hopf algebras, algebraic Lie theory, operads, and category theory.

**Encyclopedia of Mathematics and Its Applications**

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the **Encyclopedia of Mathematics and Its Applications** cover their subjects comprehensively. Less important results may be summarized as exercises at the ends of chapters. For technicalities, readers can be referred to the bibliography, which is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects.

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit

www.cambridge.org/mathematics.

123 H. Nagamochi and T. Ibaraki *Algorithmic Aspects of Graph Connectivity*
124 F. W. King *Hilbert Transforms I*
125 F. W. King *Hilbert Transforms II*
126 O. Calin and D.-C. Chang *Sub-Riemannian Geometry*
127 M. Grabisch *et al. Aggregation Functions*
128 L. W. Beineke and R. J. Wilson (eds.) with J. L. Gross and T. W. Tucker *Topics in Topological Graph Theory*
129 J. Berstel, D. Perrin and C. Reutenauer *Codes and Automata*
130 T. G. Faticoni *Modules over Endomorphism Rings*
131 H. Morimoto *Stochastic Control and Mathematical Modeling*
132 G. Schmidt *Relational Mathematics*
133 P. Kornerup and D. W. Matula Finite *Precision Number Systems and Arithmetic*
134 Y. Crama and P. L. Hammer (eds.) *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*
135 V. Berthé and M. Rigo (eds.) *Combinatorics, Automata and Number Theory*
136 A. Kristály, V. D. Rădulescu and C. Varga *Variational Principles in Mathematical Physics, Geometry, and Economics*
137 J. Berstel and C. Reutenauer *Noncommutative Rational Series with Applications*
138 B. Courcelle and J. Engelfriet *Graph Structure and Monadic Second-Order Logic*
139 M. Fiedler *Matrices and Graphs in Geometry*
140 N. Vakil *Real Analysis through Modern Infinitesimals*
141 R. B. Paris *Hadamard Expansions and Hyperasymptotic Evaluation*
142 Y. Crama and P. L. Hammer *Boolean Functions*
143 A. Arapostathis, V. S. Borkar and M. K. Ghosh *Ergodic Control of Diffusion Processes*
144 N. Caspard, B. Leclerc and B. Monjardet *Finite Ordered Sets*
145 D. Z. Arov and H. Dym *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*
146 G. Dassios *Ellipsoidal Harmonics*
147 L. W. Beineke and R. J. Wilson (eds.) with O. R. Oellermann *Topics in Structural Graph Theory*
148 L. Berlyand, A. G. Kolpakov and A. Novikov *Introduction to the Network Approximation Method for Materials Modeling*
149 M. Baake and U. Grimm *Aperiodic Order I: A Mathematical Invitation*
150 J. Borwein *et al. Lattice Sums Then and Now*
151 R. Schneider *Convex Bodies: The Brunn–Minkowski Theory (Second Edition)*
152 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions (Second Edition)*
153 D. Hofmann, G. J. Seal and W. Tholen (eds.) *Monoidal Topology*
154 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras I: The Vidav–Palmer and Gelfand–Naimark Theorems*
155 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables (Second Edition)*
156 L. W. Beineke and R. J. Wilson (eds.) with B. Toft *Topics in Chromatic Graph Theory*
157 T. Mora *Solving Polynomial Equation Systems III: Algebraic Solving*
158 T. Mora *Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond*
159 V. Berthé and M. Rigo (eds.) *Combinatorics, Words and Symbolic Dynamics*
160. B. Rubin *Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis*
161 M. Ghergu and S. D. Taliaferro *Isolated Singularities in Partial Differential Inequalities*
162 G. Molica Bisci, V. D. Radulescu and R. Servadei *Variational Methods for Nonlocal Fractional Problems*
163 S. Wagon *The Banach–Tarski Paradox (Second Edition)*
164 K. Broughan *Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents*
165 K. Broughan *Equivalents of the Riemann Hypothesis II: Analytic Equivalents*
166 M. Baake and U. Grimm (eds.) *Aperiodic Order II: Crystallography and Almost Periodicity*
167 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras II: Representation Theory and the Zel'manov Approach*
168 A. Yu. Khrennikov, S. V. Kozyrev and W. A. Zúñiga-Galindo *Ultrametric Pseudodifferential Equations and Applications*
169 S. R. Finch *Mathematical Constants II*
170 J. Krajíček *Proof Complexity*
171 D. Bulacu, S. Caenepeel, F. Panaite and F. Van Oystaeyen *Quasi-Hopf Algebras*
172 P. McMullen *Geometric Regular Polytopes*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

# *Bimonoids for Hyperplane Arrangements*

MARCELO AGUIAR
*Cornell University, Ithaca*

SWAPNEEL MAHAJAN
*Indian Institute of Technology, Bombay*

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108495806
DOI: 10.1017/9781108863117



First published 2020

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

*A catalogue record for this publication is available from the British Library.*

ISBN 978-1-108-49580-6 Hardback

# Contents

CAMBRIDGE

Cambridge University Press
978-1-108-49580-6 — Bimonoids for Hyperplane Arrangements
Marcelo Aguiar , Swapneel Mahajan
Frontmatter
More Information

# Preface

## Motivation

The geometry of braid arrangements is intimately related to the theory of connected graded Hopf algebras. Our path to this conclusion started with [**17**], where we noticed that the product and coproduct of certain combinatorial Hopf algebras are related to the geometric operations of join and link on the faces of a braid arrangement, while the compatibility axiom is related to the Tits product of those faces. This viewpoint was strengthened further when we studied Hopf monoids in Joyal species [**18**], [**19**], where the connection to the braid arrangement is more direct. This also suggested an extension of the theory to more general contexts involving real hyperplane arrangements or even certain semigroups replacing the Tits monoid of the braid arrangement. We mentioned this point for the first time in the introduction to [**19**] and then with more insistence in the end-of-chapter notes in [**21**].

## Main players

The goal of this monograph is to embark on the theory of species and bimonoids for hyperplane arrangements. The main players that have emerged in this study are summarized in Table I below.

Table I. Coxeter bimonoids and Joyal bimonoids.

| Starting data | Objects of interest | |
|---|---|---|
| hyperplane arrangement | species | |
| reflection arrangement | Coxeter species | Coxeter spaces |
| braid arrangement | Joyal species | graded vector spaces |
| hyperplane arrangement | bimonoids | |
| reflection arrangement | Coxeter bimonoids | Coxeter bialgebras |
| braid arrangement | Joyal bimonoids | graded bialgebras |

Our hyperplane arrangements are assumed to be linear, that is, all hyperplanes pass through the origin. We will use the term 'classical' to refer to the mathematics of the braid arrangement. In most cases, classical objects will mean graded vector spaces and graded bialgebras, but they can also mean Joyal species and Joyal bimonoids.

In this book, we focus exclusively on species and bimonoids for which the starting data is a hyperplane arrangement. We also briefly indicate how they relate to Joyal species and Joyal bimonoids. Analogies with connected graded Hopf algebras are mentioned but not elaborated. In fact, these classical objects are better connected to Coxeter bimonoids and Coxeter bialgebras which are more structured and for which the starting data is a reflection arrangement. These Coxeter objects along with their relation to the classical theory will be explained in a separate work.

Table II. Coxeter operads and May operads.

| Starting data | Objects of interest |
|---|---|
| Hyperplane arrangement | Operads |
| Reflection arrangement | Coxeter operads |
| Braid arrangement | May operads |

Related objects are summarized in Table II. In this book, we briefly develop operads in the setting of hyperplane arrangements, and explain their connection to species and bimonoids. We also indicate how they relate to classical operads (which we call May operads). A proper understanding of this relationship requires consideration of Coxeter operads which will be treated in a separate work.

As a historical note, we mention that the picture in Tables I and II along with all the basic definitions emerged as [**18**] was nearing publication and became completely clear to us by the time [**19**] was published.

### Synopsis

We begin by introducing the category of species for any hyperplane arrangement and the notions of monoid, comonoid, bimonoid therein. These may be viewed as an extension of corresponding notions in Joyal species from braid arrangements to an arbitrary arrangement. The main novelty is the usage of the Tits product on faces in the formulation of the bimonoid axiom. (We use the term ‘bimonoid’ rather than ‘Hopf monoid’ since we only treat the connected case.) A bimonoid can be commutative, cocommutative, both or neither. Illustrative examples include the exponential bimonoid, the bimonoid of chambers, the bimonoid of flats, the bimonoid of faces, the bimonoid of top-nested faces, the bimonoid of top-lunes, the bimonoid of bifaces, the bimonoid of chamber maps and the bimonoid of pairs of chambers. We also define signed bimonoids. More generally, for any scalar $q$, we introduce $q$-bimonoids, with $q = \pm 1$ specializing to bimonoids and signed bimonoids, respectively. This is done by deforming the bimonoid axiom using the distance function on faces of the arrangement. We also introduce the notion of a Lie monoid for any hyperplane arrangement.

We briefly consider operads in the setting of hyperplane arrangements. These may be seen as an extension of May operads from braid arrangements to an arbitrary arrangement. We define the commutative operad, associative operad, Lie operad for any hyperplane arrangement, and observe that left modules over these operads in the category of species are commutative monoids, monoids, Lie monoids, respectively. Any operad gives rise to a monad on species with operad-modules corresponding to monad-algebras. Thus, commutative monoids, monoids, Lie monoids can also be viewed as algebras over suitable monads. Moreover, we construct bimonads (mixed distributive laws) whose bialgebras are precisely bimonoids and their commutative and signed counterparts.

We lay out the basic theory of bimonoids for hyperplane arrangements. This includes a detailed discussion of primitive filtrations of comonoids and decomposable filtrations of monoids, the related Browder–Sweedler commutativity result and Milnor–Moore cocommutativity result, universal constructions of bimonoids, the Hadamard product of bimonoids and its freeness properties, the universal measuring comonoid and enrichment of the category of monoids over the category of comonoids, the antipode of a bimonoid and the Takeuchi formula. This is largely motivated by the classical theory of Hopf algebras [**867**] and the theory of Hopf monoids in Joyal species [**18**], [**19**]. The universal constructions, for instance, employ generalizations of the classical notions of (de)shuffles and (de)quasishuffles to arrangements which were given in [**17**]. We use (noncommutative) zeta and Möbius functions introduced in [**21**] to generalize the classical exponential and logarithm and obtain a family of exp-log correspondences between (co)derivations and (co)monoid morphisms, and between primitive and group-like series of a bimonoid. We consider (commutative, two-sided) characteristic operations on bimonoids and employ them to forge a precise connection of bimonoids and their commutative counterparts with the representation theory of the Birkhoff algebra, Tits algebra, Janus algebra. These algebras appear prominently in the recent semigroup literature; see for instance [**21**]. Characteristic operations by complete systems of primitive idempotents of the Tits algebra extend the classical theory of eulerian idempotents to arrangements.

We treat in detail many important structure results for bimonoids. These are analogues of well-known classical results for Hopf algebras. Their extension to arrangements appears here for the first time and contains many new ideas. These include the Loday–Ronco theorem for 0-bimonoids, the Leray–Samelson theorem for bicommutative bimonoids, the Borel–Hopf theorem for commutative bimonoids and for cocommutative bimonoids. We also generalize the Loday–Ronco theorem to $q$-bimonoids for $q$ not a root of unity. This makes use of a classical factorization result of Varchenko on distance functions, and a $q$-analogue of zeta and Möbius functions and the resulting $q$-exp-log correspondence involving the $q$-exponential and $q$-logarithm. We treat the Poincaré–Birkhoff–Witt (PBW) and Cartier–Milnor–Moore (CMM) theorems relating Lie monoids and cocommutative bimonoids, as well as their dual versions relating Lie comonoids and commutative bimonoids, and highlight their connection with the Borel–Hopf theorem. These results come in two flavors, namely, unsigned and signed, with the two linked by the signature functor. We establish the Hoffman–Newman–Radford rigidity theorems which relate (de)shuffles and (de)quasishuffles in the setting of arrangements and are significant for the theory of zeta and Möbius functions. All our results are valid over a field of arbitrary characteristic.

### Prerequisites

The prerequisites for reading this book pertain to three main areas: category theory, Hopf and Lie theory, and hyperplane arrangements. They are elaborated below.

**Category theory.** We assume familiarity with the basic language of category theory at the level of [**54**], [**591**], [**781**], [**785**]. Some concepts which are repeatedly used without explanation are functors, natural transformations, equivalences between categories, adjunctions, universal properties. Appendices are provided for more advanced concepts such as monads.

**Hopf theory and Lie theory.** While the entire theory here is developed from first principles, some exposure to classical Hopf theory at the level of [**867**] will be useful for motivational purposes. For Lie theory we require much less, basic familiarity with Lie algebras including the construction of the universal enveloping algebra is more than sufficient. We provide ample references to the classical literature. We point out that our monograph on species and Hopf algebras [**18**] is *not* a formal prerequisite for reading this work, though again some familiarity will be useful.

**Hyperplane arrangements.** General familiarity with hyperplane arrangements is sufficient to get started. The Tits monoid is a central object; familiarity with the Tits product and its basic properties suffices to understand large parts of the text. In certain places, we do need access to more specialized notions and results. These involve incidence algebras and noncommutative zeta and Möbius functions, the structure theory of the Tits algebra, distance functions, Lie and Zie elements, the descent, lune, Witt identities. To keep the book self-contained, this material is reviewed here in an introductory chapter. The reader interested in more details can consult [**21**]. This reference, however, is *not* a prerequisite for reading the text. In fact, many ideas there can be motivated and understood using the Hopf perspective developed in this book.

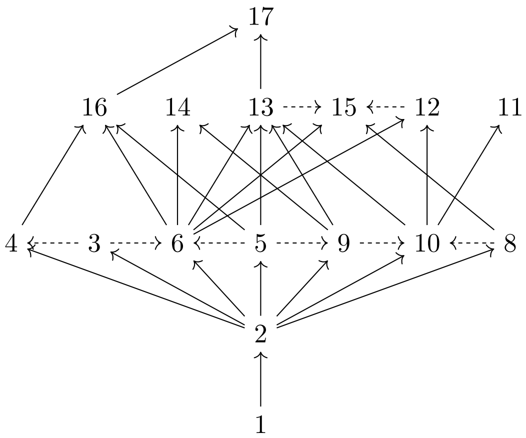
### Readership

This book would be of interest to students and researchers working in the areas of hyperplane arrangements, semigroup theory, Joyal species, May operads, Hopf algebras, algebraic Lie theory, category theory. It is written with sufficient detail to make it accessible to well-prepared graduate students.

## Organization

The text is organized in three parts. Part I introduces species and bimonoids, Part II develops their basic theory, and Part III establishes various structure results. A detailed summary of the contents is given in the main introduction. Each chapter in the text also has its own introduction. Further, there is a Notes section at the end of each chapter which provides historical commentary and detailed references to the literature. Appendices are provided for background material. Diagrams and pictures form an important component of our exposition. Numerous exercises (with generous hints) are interspersed throughout the book. We also list a few open problems. A list of notations, a list of tables, an author index, and a subject index are provided at the end of the book.

The diagram of interdependence of chapters is displayed below.




A dashed arrow indicates that the dependence is of a minimal nature. Chapter 7 is not shown in the above chart. It consists of examples and can be read in parallel to the theory developed in the other chapters. Some further guidelines on how to selectively read the book are given in the teaching section below.

## Teaching

The book is suitable for a two-semester sequence with the first semester focusing on Parts I and II, and the second on Part III. With a careful selection of topics, it can also be used for courses of shorter duration or theme-based seminars. Details follow.

(0) First do the basic groundwork, namely: Review the Tits monoid and Birkhoff monoid from Chapter 1. Define monoids, comonoids, bimonoids in species (along with their commutative counterparts) from Chapter 2. Illustrate these notions with suitable examples from Chapter 7. This could then be followed by any of the routes given below.

(1) One plan could be to do the Leray–Samelson and Borel–Hopf theorems from Chapter 13. To state these results, first review the relevant universal constructions from Chapter 6. For the proofs, three entirely different paths can be taken. The first path goes through Chapter 5 on primitive and decomposable filtrations. The second path goes through Chapter 9 on the exponential and logarithm. The third path goes through Chapters 10 and 11 related to representation theory.
(2) The pattern in item (1) could be independently followed for the Loday–Ronco and the more general rigidity theorem for $q$-bimonoids (for $q$ not a root of unity) from Chapter 13. This involves a lot of interesting $q$-calculus, with distance functions and the Varchenko factorization theorem from Chapter 1 playing a starring role.
(3) For those interested in zeta and Möbius functions, a nice option is to first do the universal constructions from Chapter 6 and then focus on the Hoffman–Newman–Radford rigidity theorems from Chapter 14. This entire chapter can be done independently of items (1) or (2), or relevant sections from it can also be done as a follow-up to either item (1) or item (2).
(4) A different plan could be to do Chapter 16 on Lie monoids. For this, first go over the commutative, associative, Lie operads and also the notion of operad modules from Chapter 4. The construction of the universal enveloping monoid requires some basic familiarity with Chapter 6. This can be followed with the Poincaré–Birkhoff–Witt theorem from Chapter 17. The final step would be to do the Cartier–Milnor–Moore theorem, but then this requires an exposure to the Borel–Hopf theorem from item (1).
(5) Independent of all the above, one can do Chapters 8 and 15 on the Hadamard product on species, OR bimonads from Chapter 3 and operads from Chapter 4 with applications to universal constructions, OR the antipode material from Chapter 12 which brings in Euler characteristics and the related descent, lune, Witt identities from Chapter 1.

Depending on interest, any of the above themes may be supplemented further in many different ways. For instance, one may discuss signed aspects of the theory, unifications via partial-flats, generalizations to left regular bands. One could also explore relevant historical facts using references in the Notes.

## Comparison with previous work

In broad terms, the text builds on [**17**], [**18**, Part II], [**19**]. However, there are some technical differences as well as some new developments which we highlight below.

**Differences.** There are two main differences to be aware of. The theory presented here is local to a fixed hyperplane arrangement, while the theory presented in [**18**] applies to not just one braid arrangement but all braid arrangements taken together. This is a local-global issue. Secondly, the tensor product of vector spaces is central to the theory of Joyal bimonoids, but that is not the case for bimonoids for arrangements. This is a noncartesian-cartesian issue. In summary, what we have developed here is a local noncartesian theory of species. This has necessitated a technical change: monoidal categories are now replaced by monads, and bilax functors now go between bimonads as opposed to between braided monoidal categories. Thus, monoids in species are now algebras over a certain monad rather than monoids in a monoidal category, and so forth.

**New developments.** The extension to arrangements brings a completely new perspective to Hopf theory. Many aspects of the theory which were implicit before have now become more visible. In addition, several new aspects of a fundamental nature have also appeared. A summary is given below.

- clarity on the central role played by the Birkhoff monoid, Tits monoid, Janus monoid,
- formulation of bimonoids and Lie monoids using 'higher operations' involving faces of the arrangement as opposed to 'binary operations' involving vertices of the arrangement,
- formulation of the commutative aspects of the theory in terms of flats, and noncommutative aspects in terms of faces,
- interpretation of the categories of monoids, comonoids, bimonoids as functor categories just like the category of species,
- emergence of many interesting finite categories constructed from geometric objects such as faces, flats, lunes, bilunes,
- connection of the antipode of a bimonoid to the antipodal map on arrangements via the antipode opposition lemma and the op-cop constructions,
- emergence of the bimonoid of bifaces and related ideas such as two-sided characteristic operations,
- connection between representation theory of the Birkhoff algebra and bicommutative bimonoids, the Tits algebra and cocommutative bimonoids, the Janus algebra and arbitrary bimonoids, and more generally, between the $q$-Janus algebra and $q$-bimonoids,
- relevance of the Karoubi envelopes of the Birkhoff monoid, Tits monoid, Janus monoid to Hopf theory,
- systematic use of distance functions and the gate property to study deformations of bimonoids,
- emergence of the monoidal category of dispecies with the category of species as a left module category over it,
- connection between operads and incidence algebras, and in particular, between the commutative operad and the flat-incidence algebra, and the associative operad and the lune-incidence algebra,
- connection of the binary quadratic presentations of the commutative and associative operads to the strong connectivity property of the posets of flats and faces, respectively,
- emergence of the one-dimensional orientation and signature spaces of an arrangement to deal with signed aspects of the theory,

- emergence of noncommutative zeta and Möbius functions as a generalization of the classical exponential and logarithm, and their intimate connection to the Hoffman–Newman–Radford rigidity theorems as well as to the Borel–Hopf theorem and Poincaré–Birkhoff–Witt theorem,
- connection of the Borel–Hopf theorem to the Zaslavsky formula for enumeration of chambers and faces in a hyperplane arrangement,
- usage of (commutative, two-sided) characteristic operations and the structure theory of the Birkhoff algebra, Tits algebra, Janus algebra to give constructive proofs of the Leray–Samelson, Borel–Hopf, Loday–Ronco theorems, respectively.

**Domain of validity.** All our results are valid over a field of arbitrary characteristic. A key reason for this is the existence of noncommutative zeta and Möbius functions over any field.

Similarly, all our results (except those of an enumerative nature or pertaining to self-duality) are valid without any finite-dimensionality assumption on the species or on the monoids, comonoids, bimonoids involved. A key reason for this is that we are working in a noncartesian setting.

**New results and topics.** Some important new topics and results are listed below.

- We establish a noncommutative analogue of the Zaslavsky formula. It involves the antipodal map on arrangements, and has connections to the Witt identities via noncommutative Möbius inversion. These ideas are also intimately tied to the logarithm of the antipode map of bimonoids.
- We introduce partially commutative monoids as interpolating objects between monoids and commutative monoids. We formalize the close parallel between the Loday–Ronco and Leray–Samelson theorems using this approach. There is a similar parallel between Borel–Hopf and Leray–Samelson.
- We prove a rigidity theorem for $q$-bimonoids when $q$ is not a root of unity. Setting $q = 0$ recovers the Loday–Ronco theorem. As a part of this story, we introduce the bilune-incidence algebra, define the two-sided $q$-zeta and $q$-Möbius functions therein, and use them to develop the $q$-exp-log correspondence. The two-sided $q$-zeta function is related to the inverse of the Varchenko matrices associated to the $q$-distance function on faces.
- We establish the Hoffman–Newman–Radford rigidity theorems. They come in different flavors; each flavor corresponds to a specific type of zeta and Möbius function. We use conjugation by the Hoffman–Newman–Radford isomorphisms to study the nondegeneracy of the mixed distributive law for bicommutative bimonoids and also for $q$-bimonoids for $q$ not a root of unity.
- We study in depth the Hadamard product on species. This includes the construction of a variety of internal homs. We introduce the bimonoid of star families built from the internal hom for comonoids, and a similar bimonoid built from the universal measuring comonoid, and explain

their connection to exp-log correspondences. We also study freeness properties of bimonoids arising as Hadamard products of bimonoids.
- We introduce the Solomon operator on the free bimonoid on a species to give a constructive proof of the Poincaré–Birkhoff–Witt theorem. We also give a novel proof of the Cartier–Milnor–Moore theorem by linking the Tits algebra to the Lie-incidence algebra. The latter is an algebra associated to the Lie operad. There is a family of isomorphisms between the two algebras indexed by noncommutative zeta and Möbius functions.

In particular, this includes progress on some questions raised in our monograph on species and Hopf algebras [**18**, Questions 12.27, 12.39, 12.67].

### Future directions

**Coxeter species and Coxeter spaces.** This monograph along with our previous work [**21**] gives a glimpse into how ideas from hyperplane arrangements and ideas from Hopf theory and algebraic Lie theory can interact and enrich each other. In a follow-up work, as mentioned in the paragraphs following Table I, we plan to:

- develop the notions of Coxeter bimonoids and Coxeter bialgebras, and the theory of Fock functors which relates the two notions,
- compare this picture with the classical picture of Joyal bimonoids, graded bialgebras and Fock functors, and
- in particular, explain how results about Coxeter bimonoids and Coxeter bialgebras can be used to deduce the corresponding results about Joyal bimonoids and graded bialgebras.

To deal with Coxeter species and Coxeter spaces, one needs to work with an invariant noncommutative zeta or Möbius function. This exists iff the field characteristic does not divide the order of the Coxeter group [**21**, Lemma 16.42]. This is how field characteristic issues eventually enter the picture.

The connection between Joyal species and species for arrangements is briefly indicated in Section 2.16 and Section 17.7. Similarly, the connection between formal power series and lune-incidence algebras is indicated in Section 9.8, see in particular Table 9.2.

**Coxeter operads.** In a similar vein, we plan to develop the notion of Coxeter operad mentioned in Table II. This will include aspects of homological algebra such as the Koszul theory of Coxeter operads, with the basic object being a differential graded Coxeter species. Given the wide applicability of May operads, this line of research looks very promising.

The connection between May operads and operads for arrangements is briefly indicated in Section 4.6.

**Semigroup theory.** Our notions of species and monoids, comonoids, bimonoids therein are defined for a fixed real hyperplane arrangement, with a central role played by the Tits monoid. These notions continue to make sense when the Tits monoid is replaced by an arbitrary left regular band (LRB) (Section 3.9) and large parts of the theory extend to this setting. For instance, see Section 4.14 for operads, Section 9.2.8 for the exp-log correspondence, Section 13.5.2 for the Leray–Samelson and Borel–Hopf theorems, Section 17.8 for the Poincaré–Birkhoff–Witt and Cartier–Milnor–Moore theorems. This line of study can lead to a better understanding of the interactions between Hopf theory and semigroup theory.

We point out a few situations where the generalization from arrangements to LRBs is not clear. There is no notion of opposite for an arbitrary LRB, so most of the descent, lune, Witt identities (Section 1.7) do not work as stated. This issue carries forward to the noncommutative Zaslavsky formula (Section 1.8) and to the antipode (Chapter 12). The orientation space and signature space do not work as stated. In particular, this affects all signed aspects of the theory such as signed commutative monoids (Section 2.5), the monad $\mathcal{E}$ (Section 3.2.6), signed Lie monoids (Section 16.7). In a similar vein, results such as rigidity of $q$-bimonoids for $q$ not a root of unity (Theorem 13.77) rely on Theorem 1.10 on Varchenko matrices which is specific to arrangements. The presentation of the Lie operad given in Example 4.12 does not hold for an arbitrary LRB. So to define LRB Lie monoids (Section 17.8), one cannot directly use the Lie bracket (generator-relation) approach (Section 16.1.2); however, the operadic approach does work (Section 16.1.1). There are similar issues with the results in Section 2.12 which are linked to the presentations of the commutative and associative operads (Examples 4.9 and 4.10).

We mention that abstract distance functions on LRBs were introduced in [**21**, Appendix E] and deserve to be studied further.

## Acknowledgements

We warmly thank our mentors Ken Brown and Steve Chase for their advice and encouragement over the years. We continue to find inspiration in the work of André Joyal, Jacques Tits, and Gian-Carlo Rota. Our gratitude goes also to Lou Billera and Persi Diaconis whose support we greatly value. We also thank Tom Leinster and Ieke Moerdijk for valuable advice regarding the publication of our work. The second author thanks all students at IIT Mumbai who attended his lectures on the subject for their valuable feedback.

Thanks to the Cambridge University Press for publishing our work in this series. Special thanks to Roger Astley for his support during the entire production process. Thanks also to Anna Scriven, Suresh Kumar and Clare Dennison.

Aguiar supported by NSF grants DMS-1001935, DMS-1401113 and Simons Foundation grants 560656 and 586087.