Introduction

We now describe the contents of the monograph in more detail along with pointers to important results. For organizational purposes, the text has been divided into three parts.

Part I

We begin Part I with a brief review of hyperplane arrangements. We then introduce the central objects, namely, species and monoids, comonoids and bimonoids for hyperplane arrangements, and initiate their basic study. We also briefly consider operads in the setting of hyperplane arrangements, and explain their connection to bimonoids. Monads and monoidal categories provide the categorical spine for these considerations.

Hyperplane arrangements. (Chapter 1.) A hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes in a real vector space. We assume that all hyperplanes pass through the origin. These hyperplanes break the space into subsets called faces. Let $\Sigma[\mathcal{A}]$ denote the set of faces. It is a graded poset under inclusion. We usually denote faces by the letters A, B, F, G, H, K. There is a unique minimum face called the central face. We denote it by O. Maximal faces are called chambers, and we denote them by the letters C, D, E. The set of faces $\Sigma[\mathcal{A}]$ is also a monoid. We call it the *Tits monoid*. The product of F and G is denoted FG and called the Tits product. Further, the product of a face and a chamber is a chamber, so the set of chambers $\Gamma[\mathcal{A}]$ is a left $\Sigma[\mathcal{A}]$-set.

Subspaces obtained by intersecting hyperplanes are called flats. Let $\Pi[\mathcal{A}]$ denote the set of flats. It is a graded lattice under inclusion. We usually denote flats by the letters X, Y, Z, W. The minimum and maximum flats are denoted $\bot$ and $\top$. The set of flats $\Pi[\mathcal{A}]$ is a commutative monoid under the join operation, that is, the product of X and Y is $\mathrm{X} \vee \mathrm{Y}$. We call this the *Birkhoff monoid*.

Every face has a support given by its linear span. It is a flat. We write $\mathrm{s}(F)$ for the support of F. The map $\mathrm{s} : \Sigma[\mathcal{A}] \to \Pi[\mathcal{A}]$ which sends F to $\mathrm{s}(F)$ is a morphism of monoids.

A biface is a pair (F, F') of faces such that F and F' have the same support. The *Janus monoid* $\mathrm{J}[\mathcal{A}]$ consists of bifaces (F, F') under the product $(F, F')(G, G') := (FG, G'F')$. It is the fiber product of the Tits monoid and its opposite monoid over the Birkhoff monoid.

The Tits algebra, Birkhoff algebra, Janus algebra are obtained from the corresponding monoids by linearization over a field $\Bbbk$. We denote them by $\Sigma[\mathcal{A}]$, $\Pi[\mathcal{A}]$, $\mathsf{J}[\mathcal{A}]$, respectively. Similarly, linearizing the set of chambers yields a left module $\Gamma[\mathcal{A}]$ over the Tits algebra. The space of chambers $\Gamma[\mathcal{A}]$ contains an important subspace, namely, the space of Lie elements which we denote by $\mathsf{Lie}[\mathcal{A}]$. Similarly, $\Sigma[\mathcal{A}]$ contains the space of Zie elements which we denote by $\mathsf{Zie}[\mathcal{A}]$.

For a flat X of $\mathcal{A}$, one can define the arrangement under X denoted $\mathcal{A}^{\mathrm{X}}$ and the arrangement over X denoted $\mathcal{A}_{\mathrm{X}}$. Further, for $\mathrm{X} \leq \mathrm{Y}$, we have the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$ obtained by first going under Y and then over X, or equivalently, by first going over X and then under Y.

We make a note of some other important algebraic objects. The *flat-incidence algebra* is the incidence algebra of the lattice of flats. It contains the zeta function ζ and Möbius function μ. The *lune-incidence algebra* is a certain reduced incidence algebra of the poset of faces. It contains noncommutative zeta functions $\boldsymbol{\zeta}$ defined by the lune-additivity formula (1.42) and noncommutative Möbius functions $\boldsymbol{\mu}$ defined by the noncommutative Weisner formula (1.44). A related object that we introduce is the *bilune-incidence algebra*. For q not a root of unity, it contains the two-sided q-zeta function $\boldsymbol{\zeta}_q$ defined by the two-sided q-lune-additivity formula (1.66) and the two-sided q-Möbius function $\boldsymbol{\mu}_q$ defined by the two-sided q-Weisner formula (1.67).

The Zaslavsky formula for chamber enumeration is recalled in (1.84). We then establish noncommutative analogues of this formula, see (1.88) and (1.89). They involve noncommutative zeta and Möbius functions.

Species. (Chapter 2.) Fix a real hyperplane arrangement $\mathcal{A}$, and a field $\Bbbk$. A *species* p is a family of $\Bbbk$-vector spaces $\mathsf{p}[F]$, one for each face F of $\mathcal{A}$, together with linear maps

$$\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G],$$

whenever F and G have the same support, such that

$$\beta_{H,F} = \beta_{H,G}\beta_{G,F} \quad \text{and} \quad \beta_{F,F} = \mathrm{id},$$

the former whenever F, G, H have the same support, and the latter for every F. (The letter β suggests a connection to braiding in monoidal categories.) A map of species $f : \mathsf{p} \to \mathsf{q}$ is a family of linear maps

$$f_F : \mathsf{p}[F] \to \mathsf{q}[F],$$

one for each face F, such that $f_G\beta_{G,F} = \beta_{G,F}f_F$ whenever F and G have the same support. This defines the category of species.

Species can also be formulated using flats instead of faces as follows (Proposition 2.5). A *species* p is a family $\mathsf{p}[\mathrm{X}]$ of $\Bbbk$-vector spaces, one for each flat X of $\mathcal{A}$. A map of species $f : \mathsf{p} \to \mathsf{q}$ is a family of linear maps $f_{\mathrm{X}} : \mathsf{p}[\mathrm{X}] \to \mathsf{q}[\mathrm{X}]$, one for each flat X.

Either formulation can be used depending on convenience of the context.

Examples of species. (Chapter 7.) The *exponential species* E is one of the most basic and important examples of a species. It is defined by setting $\mathsf{E}[\mathrm{X}] := \Bbbk$ for all flats X. Alternatively, put $\mathsf{E}[A] := \Bbbk$ for all faces A and $\beta_{B,A} = \mathrm{id}$ for all faces A and B of the same support. We mention that E has a signed analogue E^- which we call the *signed exponential species*.

The *species of flats* Π is defined by setting the component $\Pi[\mathrm{X}]$ to be the linear span of flats greater than X. For clarity, the basis element of $\Pi[\mathrm{X}]$ indexed by the flat Y is denoted $\mathtt{H}_{\mathrm{Y}/\mathrm{X}}$.

The *species of chambers* Γ is defined by setting the component $\Gamma[A]$ to be the linear span of chambers greater than A. For clarity, the basis element of $\Gamma[A]$ indexed by the chamber C is denoted $\mathtt{H}_{C/A}$. For faces A and B of the same support,

$$\beta_{B,A} : \Gamma[A] \to \Gamma[B], \qquad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{BC/B},$$

where BC denotes the Tits product of B and C.

The *species of faces* Σ is defined in a similar manner by replacing chambers by arbitrary faces. The inclusion map $\Gamma \hookrightarrow \Sigma$ is a map of species.

Many more examples are discussed in the text such as the species of charts, top-nested faces, top-lunes, bifaces, and so on.

Bimonoids. (Chapters 2 and 7.) A *monoid*, denoted (a, μ), is a species a equipped with linear maps

$$\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A],$$

one for each pair of faces $A \leq F$, such that

$$\mu_B^{BF} \beta_{BF,F} = \beta_{B,A} \mu_A^F, \quad \mu_A^G = \mu_A^F \mu_F^G, \quad \mu_A^A = \mathrm{id}\,.$$

These are the naturality, associativity, unitality axioms, respectively. In the naturality axiom, A and B have the same support and $A \leq F$, which implies B and BF have the same support and $B \leq BF$. In the associativity axiom, $A \leq F \leq G$. In the unitality axiom, A is an arbitrary face. We refer to μ as the *product* of a.

A *comonoid*, denoted (c, Δ), is defined dually using linear maps

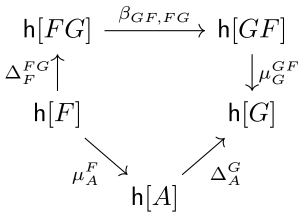
$$\Delta_A^F : \mathsf{c}[A] \to \mathsf{a}[F]$$

for $A \leq F$. We refer to Δ as the *coproduct* of c.

A *bimonoid* is a triple $(\mathsf{h}, \mu, \Delta)$, where h is a species, (h, μ) is a monoid, (h, Δ) is a comonoid, and

$$\Delta_A^G \mu_A^F = \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG}$$

for any faces $A \leq F$ and $A \leq G$. This is the *bimonoid axiom*. Note very carefully how the product of the Tits monoid enters into the axiom. The idea is to change a product followed by a coproduct to a coproduct followed by a product. However, since the Tits monoid is not commutative, $FG \neq GF$ in general. Nonetheless, FG and GF have the same support, so they can be related by β, and this intervenes in the axiom. The axiom is shown in diagrammatic form below.



More generally, for any scalar q, we define a q-bimonoid proceeding as above, but with the bimonoid axiom deformed to

$$\Delta^G_A \mu^F_A = q^{\mathrm{dist}(FG,GF)} \mu^{GF}_G \beta_{GF,FG} \Delta^{FG}_F,$$

where $\mathrm{dist}(FG, GF)$ is the number of hyperplanes which separate the faces FG and GF. Setting $q = 1$ recovers the bimonoid axiom. Other parameter values of immediate interest are $q = -1$ and $q = 0$. We use the term *signed bimonoid* to refer to a (-1)-bimonoid.

The species of chambers Γ carries the structure of a bimonoid, with product and coproduct defined by

$$\mu^F_A : \Gamma[F] \to \Gamma[A] \qquad\qquad \Delta^F_A : \Gamma[A] \to \Gamma[F]$$
$$\mathtt{H}_{C/F} \mapsto \mathtt{H}_{C/A} \qquad\qquad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{FC/F}.$$

More generally, Γ becomes a q-bimonoid if the coproduct is deformed to

$$\mathtt{H}_{C/A} \mapsto q^{\mathrm{dist}(C,FC)}\, \mathtt{H}_{FC/F}.$$

To show dependence on q, we denote it by Γ_q. We call it the q-bimonoid of chambers.

The bimonoid of faces Σ and its q-analogue can be defined in a similar manner by replacing chambers by faces. The inclusion map $\Gamma \hookrightarrow \Sigma$ is a morphism of bimonoids.

We mention in passing that one can also define the q-bimonoid of top-nested faces and the q-bimonoid of bifaces.

(Co, bi)commutative bimonoids. (Chapters 2 and 7.) A monoid (a, μ) is *commutative* if

$$\mu^F_A = \mu^G_A \beta_{G,F}$$

whenever $A \leq F$ and $A \leq G$, and F and G have the same support. This is the *commutativity axiom*. Dually, a comonoid (c, Δ) is *cocommutative* if

$$\Delta^G_A = \beta_{G,F} \Delta^F_A$$

whenever $A \leq F$ and $A \leq G$, and F and G have the same support. This is the *cocommutativity axiom*. A bimonoid can be commutative, cocommutative, both or neither. If it is both, then we use the term *bicommutative*.

There is also a notion of a *signed commutative monoid*, and dually that of a *signed cocommutative comonoid*. Moreover, the two can be combined to yield the notion of a *signed bicommutative signed bimonoid*.

The bimonoid of chambers Γ is cocommutative but not commutative. Similarly, Γ_{-1} is signed cocommutative but not signed commutative. Similar remarks apply to the bimonoid of faces Σ.

Commutativity can also be formulated directly in terms of flats as follows (Propositions 2.20, 2.21, 2.22).

A *commutative monoid*, denoted (a, μ), is a species a equipped with linear maps

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{a}[\mathrm{X}] \to \mathsf{a}[\mathrm{Z}],$$

one for each pair of flats $\mathrm{Z} \leq \mathrm{X}$, such that

$$\mu_{\mathrm{Z}}^{\mathrm{X}} = \mu_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Y}}^{\mathrm{X}} \quad \text{and} \quad \mu_{\mathrm{Z}}^{\mathrm{Z}} = \mathrm{id},$$

the former for every $\mathrm{Z} \leq \mathrm{Y} \leq \mathrm{X}$, and the latter for every Z. These are the associativity and unitality axioms, respectively. A morphism of commutative monoids $f : \mathsf{a} \to \mathsf{b}$ is a map of species f such that $f_{\mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} = \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}}$ for every $\mathrm{Z} \leq \mathrm{X}$. This defines the category of commutative monoids.

A *cocommutative comonoid*, denoted (c, Δ), is defined dually using linear maps

$$\Delta_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{c}[\mathrm{Z}] \to \mathsf{c}[\mathrm{X}]$$

for $\mathrm{Z} \leq \mathrm{X}$.

A *bicommutative bimonoid* is a triple $(\mathsf{h}, \mu, \Delta)$, where h is a species, (h, μ) is a commutative monoid, (h, Δ) is a cocommutative comonoid, and

$$\Delta_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Z}}^{\mathrm{X}} = \mu_{\mathrm{Y}}^{\mathrm{X} \vee \mathrm{Y}} \Delta_{\mathrm{X}}^{\mathrm{X} \vee \mathrm{Y}}$$

for any flats $\mathrm{Z} \leq \mathrm{X}$ and $\mathrm{Z} \leq \mathrm{Y}$. This is the *bicommutative bimonoid axiom*. It allows us to change a product followed by a coproduct to a coproduct followed by a product. Note very carefully how the product of the Birkhoff monoid (join operation on flats) has entered into the axiom. The axiom is shown in diagrammatic form below.

$$\begin{array}{ccccc}
 & \Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{Y}} & \mathsf{h}[\mathrm{X} \vee \mathrm{Y}] & \mu_{\mathrm{Y}}^{\mathrm{Y}\vee\mathrm{X}} & \\
 & \nearrow & & \searrow & \\
\mathsf{h}[\mathrm{X}] & & & & \mathsf{h}[\mathrm{Y}] \\
 & \searrow & & \nearrow & \\
 & \mu_{\mathrm{Z}}^{\mathrm{X}} & \mathsf{h}[\mathrm{Z}] & \Delta_{\mathrm{Z}}^{\mathrm{Y}} &
\end{array}$$

A morphism of bicommutative bimonoids is a map of the underlying species such that $f_{\mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} = \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}}$ and $f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}} = \Delta_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{Z}}$ for $\mathrm{Z} \leq \mathrm{X}$.

The exponential species E is a bicommutative bimonoid with $\mu_{\mathrm{Z}}^{\mathrm{X}} = \mathrm{id}$ and $\Delta_{\mathrm{Z}}^{\mathrm{X}} = \mathrm{id}$. We mention that the signed exponential species E^{-} carries the structure of a signed bicommutative signed bimonoid.

The species of flats Π is a bicommutative bimonoid, with product and coproduct defined by

$$\mu_{\mathrm{Z}}^{\mathrm{Y}} : \Pi[\mathrm{Y}] \to \Pi[\mathrm{Z}] \qquad \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \Pi[\mathrm{Z}] \to \Pi[\mathrm{Y}]$$

$$\mathtt{H}_{\mathrm{X}/\mathrm{Y}} \mapsto \mathtt{H}_{\mathrm{X}/\mathrm{Z}} \qquad \mathtt{H}_{\mathrm{X}/\mathrm{Z}} \mapsto \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{Y}}.$$

Duality. (Chapter 2.) The duality operation on vector spaces extends to species. The dual p^* of a species p is defined by $\mathsf{p}^*[\mathrm{X}] := \mathsf{p}[\mathrm{X}]^*$. Duality interchanges (commutative) monoids and (cocommutative) comonoids, and preserves bimonoids. Thus, if h is a bimonoid, then so is h^*.

A bimonoid is *self-dual* if it is isomorphic to its own dual. For instance, E and Π are self-dual, but Γ and Σ are not (since Γ is cocommutative but not commutative, while Γ^* shows the opposite behavior).

Monads. (Chapter 3.) (Co, bi)monads on a category and (co, bi)lax functors linking them are reviewed in Appendix C. We recall from Definition C.4 that a bimonad is a triple $(\mathcal{V},\mathcal{U},\lambda)$ consisting of a monad $\mathcal{V}$, a comonad $\mathcal{U}$, and a mixed distributive law $\lambda : \mathcal{V}\mathcal{U} \to \mathcal{U}\mathcal{V}$ linking them. There are also notions of monad algebra, comonad coalgebra and bimonad bialgebra.

We construct a bimonad on species which we denote by $(\mathcal{T},\mathcal{T}^\vee,\lambda)$ (Theorem 3.4). A $\mathcal{T}$-algebra is the same as a monoid, a $\mathcal{T}^\vee$-coalgebra is the same as a comonoid, and a $(\mathcal{T},\mathcal{T}^\vee,\lambda)$-bialgebra is the same as a bimonoid (Proposition 3.5). Moreover, for any scalar q, one can deform the mixed distributive law λ to λ_q such that the resulting bialgebras are q-bimonoids (Theorem 3.6).

Similarly, we construct another bimonad on species denoted $(\mathcal{S},\mathcal{S}^\vee,\lambda)$. A $\mathcal{S}$-algebra is the same as a commutative monoid, a $\mathcal{S}^\vee$-coalgebra is the same as a cocommutative comonoid, and a $(\mathcal{S},\mathcal{S}^\vee,\lambda)$-bialgebra is the same as a bicommutative bimonoid. The precise connection between this bimonad and the previous one is summarized in Proposition 3.13.

Operads. (Chapter 4.) A *dispecies* $\mathbf{p}$ is a family $\mathbf{p}[\mathrm{X},\mathrm{Y}]$ of $\Bbbk$-vector spaces, one for each pair (X,Y) of flats with $\mathrm{X} \leq \mathrm{Y}$. The category of dispecies carries a monoidal structure. For dispecies $\mathbf{p}$ and $\mathbf{q}$, define the dispecies $\mathbf{p} \circ \mathbf{q}$ by

$$(\mathbf{p} \circ \mathbf{q})[\mathrm{X},\mathrm{Z}] := \bigoplus_{\mathrm{Y}:\, \mathrm{X}\leq\mathrm{Y}\leq\mathrm{Z}} \mathbf{p}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{q}[\mathrm{Y},\mathrm{Z}].$$

We refer to this operation as the *substitution product* of $\mathbf{p}$ and $\mathbf{q}$. The unit object is the dispecies $\mathbf{x}$ defined by $\mathbf{x}[\mathrm{X},\mathrm{Y}] = \Bbbk$ when $\mathrm{X} = \mathrm{Y}$, and 0 otherwise.

A monoid in this monoidal category is an *operad*. Explicitly, an operad is a dispecies $\mathbf{a}$ equipped with linear maps

$$\gamma : \mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}] \to \mathbf{a}[\mathrm{X},\mathrm{Z}] \quad \text{and} \quad \eta : \Bbbk \to \mathbf{a}[\mathrm{X},\mathrm{X}],$$

the former for each $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$ and the latter for each X, subject to associativity and unitality axioms.

The category of species is a left module category over the category of dispecies as follows. For a dispecies $\mathbf{p}$ and a species m, define the species $\mathbf{p} \circ \mathsf{m}$ by

$$(\mathbf{p} \circ \mathsf{m})[\mathrm{X}] := \bigoplus_{\mathrm{Y}:\, \mathrm{Y}\geq\mathrm{X}} \mathbf{p}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}].$$

This yields the notion of a left $\mathbf{a}$-module for any operad $\mathbf{a}$. Explicitly, a left $\mathbf{a}$-module is a species m equipped with linear maps

$$\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \to \mathsf{m}[\mathrm{X}],$$

one for each $X \leq Y$, subject to associativity and unitality axioms. The free left $\mathbf{a}$-module over a species m is given by $\mathbf{a} \circ \mathsf{m}$ (Proposition 4.23).

A comonoid in the monoidal category of dispecies is a *cooperad*. It makes sense to talk of left comodules over a cooperad. Further, there is a duality functor on dispecies which interchanges operads and cooperads (and modules and comodules). A *bioperad* is a self-dual notion. It is a triple $(\mathbf{a}, \mathbf{c}, \lambda)$ consisting of an operad $\mathbf{a}$, a cooperad $\mathbf{c}$, and a mixed distributive law $\lambda : \mathbf{a} \circ \mathbf{c} \to \mathbf{c} \circ \mathbf{a}$ linking them. By combining operad-modules and cooperad-comodules, we obtain the notion of a bioperad-bimodule.

Commutative operad and associative operad. (Chapter 4.) The *commutative operad* **Com** is defined by $\mathbf{Com}[X, Y] := \Bbbk$ for all $X \leq Y$, with structure maps γ and η being identities. A left module over **Com** is precisely a commutative monoid. The commutative operad has a signed analogue denoted $\mathbf{Com}^-$. Left modules over it are signed commutative monoids.

The *associative operad* **As** is defined as follows. For any $X \leq Y$, set $\mathbf{As}[X, Y] := \Gamma[\mathcal{A}^Y_X]$, the space of chambers of the arrangement $\mathcal{A}^Y_X$. Equivalently, it is the linear span of symbols $\mathtt{H}_{F/A}$ with $A \leq F$, $s(A) = X$ and $s(F) = Y$, subject to the relation $\mathtt{H}_{F/A} = \mathtt{H}_{BF/B}$, whenever A and B have the same support. The structure map is given by

$$\gamma : \mathbf{As}[X, Y] \otimes \mathbf{As}[Y, Z] \to \mathbf{As}[X, Z], \qquad \mathtt{H}_{F/A} \otimes \mathtt{H}_{G/F} \mapsto \mathtt{H}_{G/A},$$

where A, F, G are faces with support X, Y, Z, respectively, and $A \leq F \leq G$. A left module over **As** is precisely a monoid, with $\mathtt{H}_{F/A}$ corresponding to the product component μ^F_A of the monoid.

Dualizing the commutative and associative operads yields the cooperads $\mathbf{Com}^*$ and $\mathbf{As}^*$. Left comodules over them are cocommutative comonoids and comonoids, respectively. Further, there is a mixed distributive law λ linking **As** and $\mathbf{As}^*$ such that left bimodules over the bioperad $(\mathbf{As}, \mathbf{As}^*, \lambda)$ are precisely bimonoids. Moreover, this law can be deformed by a scalar q such that the resulting left bimodules are q-bimonoids. Similarly, there is a bioperad $(\mathbf{Com}, \mathbf{Com}^*, \lambda)$ whose left bimodules are bicommutative bimonoids.

We now tie this with the earlier discussion on bimonads. A (co, bi)operad gives rise to a (co, bi)monad on species, and moreover, left (co, bi)modules over the (co, bi)operad are the same as (co, bi)algebras over the corresponding (co, bi)monad. The point is that the bioperad $(\mathbf{As}, \mathbf{As}^*, \lambda)$ yields the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ (Theorem 4.33). Similarly, $(\mathbf{Com}, \mathbf{Com}^*, \lambda)$ yields the bimonad $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$.

Lie operad. (Chapter 4.) The *Lie operad* **Lie** is defined as a suboperad of the associative operad **As** as follows. For any $X \leq Y$, set $\mathbf{Lie}[X, Y] := \mathsf{Lie}[\mathcal{A}^Y_X]$, the space of Lie elements of the arrangement $\mathcal{A}^Y_X$. Recall that this is a subspace of $\Gamma[\mathcal{A}^Y_X]$. The point is that the operad structure of **As** restricts to these subspaces and yields a suboperad.

Quadratic operads. (Chapter 4.) The *free operad* on a dispecies $\mathbf{e}$ is given by

$$\mathcal{F}_{\circ}(\mathbf{e}) := \underset{n\geq 0}{+} \mathbf{e}^{\circ n},$$

where $\mathbf{e}^{\circ n}$ is the n-fold substitution product of $\mathbf{e}$ with itself, and $+$ denotes the coproduct in the category of dispecies. Explicitly,

$$\mathcal{F}_{\circ}(\mathbf{e})[\mathrm{X},\mathrm{Z}] = \bigoplus_{\mathrm{X}\leq \mathrm{Y}_1\leq\cdots\leq \mathrm{Y}_k\leq \mathrm{Z}} \mathbf{e}[\mathrm{X},\mathrm{Y}_1]\otimes \mathbf{e}[\mathrm{Y}_1,\mathrm{Y}_2]\otimes\cdots\otimes \mathbf{e}[\mathrm{Y}_k,\mathrm{Z}],$$

where the sum is over all multichains in the interval $[\mathrm{X},\mathrm{Z}]$.

An operad $\mathbf{a}$ is *quadratic* if it can be written as a quotient of a free operad $\mathcal{F}_{\circ}(\mathbf{e})$ by an ideal generated by a subdispecies $\mathbf{r}$ of $\mathbf{e}\circ\mathbf{e}$. We denote this by $\mathbf{a} = \langle \mathbf{e} \,|\, \mathbf{r}\rangle$. We use the term *binary quadratic* if further $\mathbf{e}[\mathrm{X},\mathrm{Y}] = 0$ unless Y covers X in the lattice of flats. For any quadratic operad, one can talk about its oriented dual which is again a quadratic operad. The commutative, associative, Lie operads are binary quadratic. For the Lie operad, antisymmetry is incorporated in $\mathbf{e}$, while the Jacobi identities are in $\mathbf{r}$. The oriented quadratic dual of the associative operad is itself, while the commutative and Lie operads are oriented quadratic duals of each other (Proposition 4.14).

Part II

In Part II, we continue the development of the basic theory of bimonoids. We discuss the primitive filtration of a comonoid and dually the decomposable filtration of a monoid. We then discuss in detail various universal constructions starting with the free monoid and cofree comonoid on a species. We study the Hadamard functor and its specialization to the signature functor. The latter sets up an equivalence between the categories of bimonoids and signed bimonoids. We develop exp-log correspondences of a bimonoid by employing noncommutative zeta and Möbius functions. We forge a precise connection of bimonoids with modules over the Birkhoff algebra, Tits algebra, Janus algebra by considering characteristic operations on bimonoids. In our setting, the antipode of a bimonoid always exists and we study it using the Takeuchi formula.

Primitive filtrations and decomposable filtrations. (Chapters 5 and 7.) Every comonoid c has a *primitive part* $\mathcal{P}(\mathsf{c})$. It is a species whose A-component consists of those elements $x \in \mathsf{c}[A]$ such that $\Delta_A^F(x) = 0$ for all $F > A$. More generally, one can define a filtration of c whose first term is $\mathcal{P}(\mathsf{c})$. This is the *primitive filtration* of c. It turns c into a filtered comonoid. Dually, every monoid a has a *decomposable part* $\mathcal{D}(\mathsf{a})$, and more generally, a decomposable filtration which turns it into a filtered monoid. We refer to $\mathcal{Q}(\mathsf{a}) := \mathsf{a}/\mathcal{D}(\mathsf{a})$ as the *indecomposable part* of a.

Just like faces and chambers, Lie and Zie elements of an arrangement give rise to the Lie species and Zie species. The primitive part of the bimonoid of chambers Γ is the Lie species (Lemma 7.64), while that of the bimonoid of faces Σ is the Zie species (Lemma 7.69). We refer to these results as the *Friedrichs criteria* for Lie and Zie elements.

A map from a species to a comonoid is a *coderivation* if it maps into the primitive part of that comonoid. Dually, a map from a monoid to a species is a *derivation* if it factors through the indecomposable part of that monoid. A (co)derivation is the same as a (co)monoid morphism with the species viewed as a (co)monoid with all its nontrivial (co)product components being 0.

For a q-bimonoid, one can consider the primitive as well as the decomposable filtrations. Both of them turn it into a filtered q-bimonoid. Thus, for either filtration, one can consider the corresponding associated graded q-bimonoid. When $q = 1$, the associated graded bimonoid wrt the primitive filtration is commutative, and wrt the decomposable filtration is cocommutative (Propositions 5.62 and 5.65). These results have a signed analogue when $q = -1$. We call these the *Browder–Sweedler commutativity result* and the *Milnor–Moore cocommutativity result*, respectively.

For a bimonoid, there is a canonical map from its primitive part to its indecomposable part, see (5.50). This map is surjective iff the bimonoid is cocommutative, injective iff the bimonoid is commutative, and bijective iff the bimonoid is bicommutative (Proposition 5.56). In particular, for the bimonoid of faces Σ, this map is surjective. As an application, we deduce the existence of special Zie elements (Exercise 7.71).

Free monoid and free commutative monoid. (Chapters 6 and 7.) For a species p, define the species $\mathcal{S}(\mathsf{p})$ by

$$\mathcal{S}(\mathsf{p})[\mathrm{Z}] := \bigoplus_{\mathrm{X}\colon \mathrm{Z}\leq \mathrm{X}} \mathsf{p}[\mathrm{X}].$$

It carries the structure of a commutative monoid: For $\mathrm{Z} \leq \mathrm{Y}$, note that the summands in $\mathcal{S}(\mathsf{p})[\mathrm{Y}]$ are all contained in $\mathcal{S}(\mathsf{p})[\mathrm{Z}]$, and we define $\mu_{\mathrm{Z}}^{\mathrm{Y}}$ to be the canonical inclusion. In fact, $\mathcal{S}(\mathsf{p})$ is the *free commutative monoid* on p. In other words, $\mathcal{S}(\mathsf{p}) = \mathbf{Com} \circ \mathsf{p}$, where **Com** is the commutative operad.

Similarly, for a species p, define the species $\mathcal{T}(\mathsf{p})$ by

$$\mathcal{T}(\mathsf{p})[A] := \bigoplus_{F\colon A\leq F} \mathsf{p}[F].$$

The map $\beta_{B,A}$ is defined by summing the maps $\beta_{BF,F}$ of the species p over all $F \geq A$. Further, $\mathcal{T}(\mathsf{p})$ is a monoid with μ_A^F defined to be the canonical inclusion. This is the *free monoid* on p. In other words, $\mathcal{T}(\mathsf{p}) = \mathbf{As} \circ \mathsf{p}$, where **As** is the associative operad.

These constructions can be extended further. Let c be a comonoid. Then the coproduct of c induces coproducts on $\mathcal{S}(\mathsf{c})$ and $\mathcal{T}(\mathsf{c})$ turning them into bimonoids. Examples include the bimonoids that we have discussed earlier, namely,

$$\mathcal{S}(\mathsf{x}) = \mathsf{E}, \quad \mathcal{T}(\mathsf{x}) = \Gamma, \quad \mathcal{S}(\mathsf{E}) = \Pi, \quad \mathcal{T}(\mathsf{E}) = \Sigma.$$

Here x denotes the species whose component $\mathsf{x}[\mathrm{Y}]$ is $\Bbbk$ if $\mathrm{Y} = \top$, and 0 otherwise. We view it as a comonoid in the only way possible with $\Delta_\top^\top = \mathrm{id}$ and the remaining coproduct components being zero.

In the case of $\mathcal{T}(\mathsf{c})$, one can do more. Its coproduct can be deformed using a scalar q such that it becomes a q-bimonoid. To show dependence on q, we write $\mathcal{T}_q(\mathsf{c})$. For instance, $\mathcal{T}_q(\mathsf{x}) = \Gamma_q$, the q-bimonoid of chambers. The universal property of $\mathcal{T}_q(\mathsf{c})$ is given in Theorem 6.6. The dual version is given in Theorem 6.13. These results can be seen as formal consequences of the existence of the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$. Related universal properties involving the primitive part functor and indecomposable part functor are given in Theorems 6.31 and 6.34, respectively.

We mention that for a species p, one can also construct the free signed commutative monoid. We denote it by $\mathcal{E}(\mathsf{p})$. When $\mathsf{p} = \mathsf{x}$, we obtain the signed exponential species E^-.

Hadamard product. (Chapter 8.) For species p and q, their *Hadamard product* $\mathsf{p} \times \mathsf{q}$ is given by

$$(\mathsf{p} \times \mathsf{q})[F] := \mathsf{p}[F] \otimes \mathsf{q}[F].$$

This defines a symmetric monoidal structure on the category of species. Let $\mathrm{hom}^\times(\mathsf{p}, \mathsf{q})$ denote its internal hom. For a comonoid c and monoid a, the species $\mathrm{hom}^\times(\mathsf{c}, \mathsf{a})$ carries the structure of a monoid, while $\mathrm{hom}^\times(\mathsf{a}, \mathsf{c})$ carries the structure of a comonoid. We refer to them as the *convolution monoid* and *coconvolution comonoid*, respectively. Combining the two constructions, for bimonoids h and k, we obtain a bimonoid $\mathrm{hom}^\times(\mathsf{h}, \mathsf{k})$. This is the *biconvolution bimonoid*. When $\mathsf{h} = \mathsf{k}$, we write $\mathrm{end}^\times(\mathsf{h})$ for $\mathrm{hom}^\times(\mathsf{h}, \mathsf{h})$. A summary of these and related objects is given in Table 8.1.

The Hadamard product gives rise to a bilax functor between bimonads (Theorem 8.4). Hence, it preserves monoids, comonoids, bimonoids. Thus, we can consider its internal hom in these categories as well. Let $\mathcal{C}(\mathsf{c}, \mathsf{d})$ denote the internal hom in the category of comonoids. When c is a cocommutative comonoid, and k is a bimonoid, $\mathcal{C}(\mathsf{c}, \mathsf{k})$ carries the structure of a bimonoid. We refer to $\mathcal{C}(\mathsf{c}, \mathsf{k})$ as the *bimonoid of star families* (Section 8.4). If, in addition, c carries the structure of a bimonoid, then $\mathcal{C}(\mathsf{c}, \mathsf{k})$ can be realized as a subbimonoid of the biconvolution bimonoid $\mathrm{hom}^\times(\mathsf{c}, \mathsf{k})$ (Lemma 8.36). There is a similar bimonoid $\overline{\mathcal{C}}(\mathsf{h}, \mathsf{a})$ associated to a commutative monoid a and bimonoid h which is built out of the *universal measuring comonoid* (Section 8.6). The latter allows us to enrich the category of monoids over the category of comonoids. We describe the power and copower of this enriched category (Propositions 8.65 and 8.67).

For any species p, we let $\mathsf{p}^- := \mathsf{p} \times \mathsf{E}^-$, where E^- is the signed exponential species. We refer to p^- as the *signed partner* of p. This yields the *signature functor* which sends a species to its signed partner. It induces an adjoint equivalence between the categories of q-bimonoids and $(-q)$-bimonoids for any scalar q (Corollary 8.92).

Exp-log correspondences. (Chapter 9.) The lune-incidence algebra acts on the space of all maps from a comonoid c to a monoid a. We refer to the action of a noncommutative zeta function $\boldsymbol{\zeta}$ as an exponential, and to the action of a noncommutative Möbius functions $\boldsymbol{\mu}$ as a logarithm. This sets up exp-log correspondences on this space of maps (Proposition 9.9). Moreover, any logarithm of a comonoid morphism from a cocommutative comonoid to a