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## GENERATORS OF MARKOV CHAINS

Elementary treatments of Markov chains, especially those devoted to discrete-time and finite state-space theory, leave the impression that everything is smooth and easy to understand. This exposition of the works of Kolmogorov, Feller, Chung, Kato, and other mathematical luminaries, which focuses on time-continuous chains but is not so far from being elementary itself, reminds us again that the impression is false: an infinite, but denumerable, state-space is where the fun begins.

If you have not heard of Blackwell's example (in which all states are instantaneous), do not understand what the minimal process is, or do not know what happens after explosion, dive right in. But beware lest you are enchanted: 'there are more spells than your commonplace magicians ever dreamed of.'

**Adam Bobrowski** is Professor and Chairman of the Department of Mathematics at Lublin University of Technology, Poland. He is a pure mathematician who uses the language of operator semigroups to describe stochastic processes. He has authored and coauthored nearly 70 papers on the subject and five books, including *Functional Analysis for Probability and Stochastic Processes* (2005) and *Convergence of One-Parameter Operator Semigroups* (2016).

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'Fascinating Markov Chains' by Marek Bobrowski. In a different version of this cartoon by W. Chojnacki, the role of the anonymous examiner is played by an antagonist of A. A. Markov, that is, Pavel Niekrasov, who says, 'Contrary to my beliefs, these chains are really fascinating!' See [50] for the entire story.

# Generators of Markov Chains

From a Walk in the Interior to a Dance on the Boundary

ADAM BOBROWSKI  
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To my dancing Beatka

*I met a man once . . . to whom Heyst exclaimed, in no connection with anything in particular (it was in the billiard-room of the club): 'I am enchanted with these islands!' He shot it out suddenly, a propos des bottes, as the French say, and while chalking his cue. And perhaps it was some sort of enchantment. There are more spells than your commonplace magicians ever dreamed of.*

J. Conrad (J. T. K. Korzeniowski), *Victory*

*Nie sprawiłeś mi zawodu, synu. Przeciwnie, zadziwiłeś mnie. Zdołałeś dać z siebie więcej, niżem od ciebie oczekiwał.*

Teodor Parnicki, *Srebrne orły*

*Markov chains merely walk in their regular state space, but on the cliffs of their boundaries, they dance.*

Johann Gottfried von Spacerniak



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## Preface

*Application to business is the root of prosperity, but those who ask questions that do not concern them are steering the ship of folly towards the rock of indigence.*  
Arsheesh the greedy fisherman in C. S. Lewis's *The Horse and His Boy*

The theory of Markov chains, whether time discrete or time continuous, is one of the integral parts of the theory of stochastic processes. This book, however, is not devoted to the popular part of this rich theory, so the reader will not learn here about recurrent and transient states, ergodic theorems, or convergence to equilibrium. (In Arsheesh's words, thus, we will ask questions that do not concern us, having no business in mind, in the hope that we will somehow reach Narnia and the North.) Instead, we focus on the equally intriguing question of how a continuous-time Markov chain may be described by means of its Kolmogorov (intensity) matrix or its generator, and we study the interplay between the notions just named. We argue in particular that, despite their popularity, Kolmogorov (intensity) matrices are less suitable for such description than generators. Whereas, in their relative simplicity, they allow an intuitive formulation of processes, in general, they fail to describe more delicate phenomena.

Therefore, in Chapter 2, we compare these two notions in the light of two examples due to Kolmogorov, Kendall, and Reuter. These examples show that whereas the intensity matrix determines in a sense the way the generator acts, it may not determine the generator's domain, and without information on the shape of the domain, a Markov chain is not completely specified. Furthermore, in Chapters 3 and 4, devoted to boundary theory, we show that an explosive intensity matrix characterizes the chain only locally, up to a time of explosion. Put otherwise, the matrix characterizes merely the minimal chain, which

after explosion is undefined. There are, however, infinitely many postexplosion processes that dominate the minimal chain. Their generators may differ from the generator of the minimal chain by extra terms and may have different domains; both the domain and the terms contain crucial information on the postexplosion process; by nature, this information cannot be found in the intensity matrix. Chapter 5 presents a similar view from the dual perspective.

\* \* \*

The way information on boundary behavior of a Markov chain is reflected in its generator is a recurring subject in this book. It transpires that in  $l^1$ , the exit boundary introduces additional terms in the generator, whereas the entrance boundary affects its domain. But in  $l^\infty$ , these things are turned upside down: roughly, the exit boundary always perturbs the domain, but the entrance boundary may either introduce new terms or perturb the domain. The fact that in passing from a space to its dual a perturbation of the way a generator acts may become a perturbation of the domain, and vice versa, has been observed also in other contexts (see, e.g., [17] or Chapter 50 in [16]), but in the case of Markov chains, this phenomenon seems to be particularly intriguing and spectacular.

\* \* \*

Another idea, perhaps borrowed from [39], that permeates the book is that instead of describing extraordinary Markov chains by rigorous, but involved, stochastic constructions, stochastic intuitions may be developed by approximating this chain's semigroup of transition probabilities by a sequence of semigroups of transition probabilities related to some finite-space chains. (The theory of semigroups of operators and suitable convergence theorems are presented in Chapter 1.) Since the latter chains have considerably simpler, well-understood structure, this idea works well. This is in particular how an insight is gained into Blackwell's and both Kolmogorov–Kendall–Reuter examples. This is also how the discrete boundary for an explosive Markov chain is introduced.

\* \* \*

Yet another characteristic of the book is that we (the reader and I) allow ourselves the comfort of discovering new results gradually, step by step, not trying to reach the mountain peak immediately via the most efficient route. The proof of an introductory result may thus be more complicated than that of a general

theorem, presented later; in proving the former, we are simply yet not so clear about the general view. Neither are we afraid to spend some time looking at an illustrative example, which perhaps involves more calculations than one could wish to go through, before the idea of a general theorem following it dawns on us.

For instance, it takes several pages of investigating a particular (pure birth) Markov chain before formula (3.36) is discovered, and yet another couple of pages before an analogous formula (3.39) is derived. It is only then that the much more general formula (3.55), built by analogy to (3.36) and (3.39), is discussed, and the proof that the operator in (3.55) is a Markov generator is less than one page long (see also Section 3.5.5). Similarly, the two-pages-long proof of the master theorem of Section 3.7.9 is preceded by the more than four-pages-long discussion of a particular ‘two infinite ladder’ example of a Markov chain (see Sections 3.5.6–3.5.9).

I still think the intuitive should go before the abstract and that discovering is more fun than learning: Sections 3.5.5 and 3.7.9 occupy a special place in this book and are worth being understood thoroughly. Besides, the general idea is to enjoy ourselves as much as possible.

\* \* \*

A couple of words are due on the way boundary theory for Markov chains is presented in this book. The primary sources of information on this theory are W. Feller [41, 42] and K. L. Chung [22]. My presentation is closer to Feller’s, and draws on his heavily, but the two still differ significantly. First of all, Feller focuses on the Laplace transform of transition probabilities (of postexplosion processes), that is, on the resolvents of generators, whereas – at least to my taste – description of generators themselves is more appealing. In characterizing the latter, certain functionals show up naturally as building blocks for the generators, and thus it is the set of these functionals that is defined as the boundary. It is only later, as a sort of afterthought, that I identify these functionals as sojourn solutions for the related minimal chain. For Feller it was the latter that were the starting point of his investigations.

\* \* \*

Besides one place in Chapter 5 (Section 5.8.4), the text is self-contained, but some basic knowledge of real analysis (including Laplace transform), probability theory, and functional analysis is needed for its understanding. I assumed the reader is familiar with random variables, Banach spaces, linear operators

and their norms, and so on. Nevertheless, only elementary properties of these objects are used without explanation: I have made a definite effort to make the text as reader-friendly as possible.

\* \* \*

I would like to thank the members of the Lublin University of Technology mathematical research seminar (Adam Gregosiewicz, Andrzej Łagodowski, Iwona Malinowska, Małgorzata Murat, Ernest Nieznaj, Elżbieta Ratajczyk, Renata Rososzczuk, Katarzyna Steliga, and Łukasz Stępień) for their careful study of early drafts of this book, in their various forms, and for spotting a number of places that needed smaller and greater corrections. Tomasz Szarek also read through the entire manuscript, in just two months, and suggested several improvements; I am grateful to him for this. A number of invaluable corrections I owe, furthermore, to Marta Tyran–Kamińska. Special thanks go to Jacek Banasiak, Wojciech Chojnacki, Ryszard Rudnicki, and Yuri Tomilov for their encouragement and advice. It was by reading N. H. Bingham’s interview with David Kendall [12], suggested to me by Yura, that I became aware of Kendall and Reuter’s work on Markov chains, dots started to connect, and the idea of the present book dawned on me. I am also indebted to my physicians, in particular, to Drs. Andrzej Kwiecień, Krzysztof Paprota, and Justyna Biegańska-Siczek for keeping me alive (and kicking) while I was writing this book. Finally, I would like to thank Clare Dennison and Roger Astley from Cambridge University Press for their professional support and advice before and during the process of writing and producing this book.

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## A Nontechnical Introduction

Markov chains are one of the simplest stochastic processes. In the archetypical example of a Markov chain, that is, in the (symmetric) simple random walk on the set of integers, a traveler or a particle starting at  $i$  makes a step to the right, to  $i + 1$ , with probability  $\frac{1}{2}$  or a step to the left, to  $i - 1$ , with the same probability. At its new position (either  $i - 1$  or  $i + 1$ ), it continues in the same fashion, forgetting the past and the way it reached this position.

Amazingly, many quantitative results may be proved about such apparently completely chaotic movement. For example, a particle starting at any point will surely reach 0 at a certain time in the future. What is probably less obvious is that the same is true about an analogous random walk in two dimensions, where a particle may go to the left, to the right, up, or down. However, in three and more directions, the situation is quite different: the probability of reaching 0 from a nonzero state  $i$  is smaller than 1, and so is the probability of reaching any state different from  $i$  (see, e.g., [35]).

What distinguishes Markov chains from other, more complex stochastic processes is the state-space, that is, the set of possible states. For Markov chains, the state-space is by definition denumerable: it is either finite or there is a one-to-one correspondence between its elements and the elements of the set of positive integers. For example, the set of integers may be arranged in a sequence, and so may be the state-space of the simple random walk in any dimension.

Of course, as a result of such a rearrangement, natural neighbors of a point will probably lie quite a distance apart from each other and from the point, and the description of the walk will become less intuitive. But we are naturally led to a more general object: in a chain that is more general than the simple random walk, a particle starting at  $i$  may go to any other point  $j$  of the state-space (or stay in  $i$ ), and it does that with probability characteristic to that point. In fact, that probability, denoted  $p_{i,j}$  (and termed ‘transition probability’), in general

depends on both  $i$  and  $j$ . The only restriction is that in the honest Markov chains, the sum of  $p_{i,j}$  over all  $j$  is 1 for all  $i$ .

In this much more general setting, still much may be said of the nature of the process: for example, under reasonable assumptions on the probabilities  $p_{i,j}$ , existence of stationary distributions may be proved. (Informally speaking, stationary distributions are arrangements of collections of a large number of particles with the property that a random displacement of these particles according to the rules provided by  $p_{i,j}$ 's, though changing positions of many or all individual particles, does not change the arrangement as a whole.) Conditions are also known under which any other arrangement of particles, with displacement rules hidden in  $p_{i,j}$ , will in time become more and more close to the stationary distribution, even to the point of being practically indistinguishable, and we can even estimate the speed with which this happens [72, 78].

Everything said above concerns Markov chains in discrete time: in a discrete-time Markov chain, as described above, all changes occur at multiples of a unit time. However, there are also time-continuous Markov chains. Whereas in a discrete-time Markov chain, everything hinges on the probabilities  $p_{i,j}$ , to describe a continuous-time Markov chain, one needs intensities of jumps. In the simplest case of right-continuous processes with lefthand limits, if the intensity of a jump from a state  $i$  is  $q_i > 0$ , the process stays at  $i$  for an exponential time  $T$  so that  $\mathbb{P}(T > t) = e^{-q_i t}$ ,  $t \geq 0$ . (If  $q_i = 0$ , the process stays at  $i$  forever.) Then, it jumps to a different state  $j$ , and the probability that a particular  $j$  is chosen is  $p_{i,j} = q_{i,j}/q_i$ , where  $q_{i,j}$  is a (given) intensity of jump from  $i$  to  $j$ . The condition that the sum of  $p_{i,j}$  over all  $j$  should be 1 tells us that we should have  $\sum_{j \neq i} q_{i,j} = q_i$ . At  $j$ , the process forgets its past and continues the same procedure, with  $q_i$  replaced by  $q_j$ .

Even from this picture, it is somewhat clear that continuous-time Markov chains are in many aspects similar to discrete-time Markov chains. In the case where the state-space is finite, they indeed are much the same, and the description in terms of intensities is both appealing and useful. If the state-space is infinite (but denumerable), however, unexpected difficulties and curious phenomena might occur. First of all, some and even all states might be instantaneous, which is to say that all intensities might be infinite. Upon arriving at such a state, the process leaves 'immediately,' and the description given above does not apply, or at least does not tell the entire story. Moreover, if worst comes to worst and the times before consecutive jumps are dramatically shorter and shorter, the definition in terms of jumps may turn out inadequate, leaving the process undefined after a finite random time (being equal to the infinite sum of shorter and shorter times before consecutive jumps). In other words,



intensities of jumps do not contain the information on what happens with the process after such ‘explosion.’ On top of that, there can be many ways such explosion occurs and many, trivial and nontrivial, ways the process returns to the state-space.<sup>1</sup>

In such cases, to describe the process in full, instead of the set, or a matrix, of intensities, one needs a more complex object: a closed operator in the Banach space of absolutely summable sequences, the generator of a related semigroup of transition matrices in this space. In this book, we discuss when and in what sense such generators may be identified with matrices of jump intensities and how they can be constructed and employed when intensity matrices are seen to be less useful. In particular, we explain how the information on postexplosion behavior is built into the generators. Roughly speaking, this is done either by modifying the way the generator of the minimal chain acts (the minimal chain is the chain undefined after explosion) or by modifying the generator’s domain.

<sup>1</sup> Each way explosion occurs is a set (or, better, an equivalence class of sets) where the process ‘sojourns,’ and each such sojourn set may be seen as an additional point of the state-space for the process, a point of its exit boundary. Similarly, each nontrivial way the process may return to the state-space is a point of its entry boundary.

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