1

A Guided Tour through the Land of Operator Semigroups

1.1 Semigroups and Generators

This introductory chapter is meant as a gentle, short introduction to the theory of operator semigroups. From the beginning, this theory has been devised and seen, especially by W. Feller and E. B. Dynkin, as an efficient tool for describing Markov processes. Today – notwithstanding advances of stochastic analysis we have witnessed over the last several decades – its usefulness is still undeniable.

We present all the relevant facts of the theory, needed for understanding the main body of the book, but – for brevity – refrain from presenting detailed proofs of the main theorems on generation, perturbation, approximation and convergence of semigroups. (The almost self-contained Section 1.5, devoted to sun-dual semigroups, is an exception to this rule.) Instead, we explain all the needed notions and illustrate the theory with simple, but illuminating examples. We hope that the reader who has not come across the theory of semigroups of operators yet, and decides to take this guided tour will be well prepared for studying the rest of the book, and, if necessary, to study more advanced books on semigroups. A list of monographs containing all the missing proofs – and much more – is given in Section 1.7.

Here and there in this chapter, we will be guided by intuitions from stochastic processes and Markov chains in particular, even though the latter will not be defined before Chapter 2.

1.1.1 *Strongly continuous semigroups of operators*

A C_0 semigroup or a **strongly continuous semigroup** in a Banach space $\mathbb{X}$ is a family $T = \{T(t), t \geq 0\}$ (written also as $(T(t))_{t\geq 0}$ or $\{T_t, t \geq 0\}$) of bounded linear operators such that

(a) $T(t)T(s) = T(t+s)$, $t, s \ge 0$,
(b) $T(0) = I_{\mathbb{X}}$ (the identity operator in $\mathbb{X}$),
(c) $\lim_{t\to 0} T(t)x = x$, $x \in \mathbb{X}$,

with the last limit in the norm of $\mathbb{X}$. Condition (a) is termed the **semigroup property**. If merely the first two conditions are satisfied, the family T is said to be a **semigroup**.

The following intuition is hidden behind this notion. Throughout large parts of this book, $\mathbb{X}$ will be the space

$$l^1 = l^1(\mathbb{I})$$

of absolutely summable sequences $x = (\xi_i)_{i\in\mathbb{I}}$ indexed by elements of a countable set $\mathbb{I}$. (Recall that the norm in this space is $\|x\| = \sum_{i\in\mathbb{I}} |\xi_i|$.) Nonnegative elements of l^1 with coordinates summing to 1 can be thought of as initial distributions of an underlying Markov chain, and then, in many cases of interest to us, $T(t)$ can be interpreted as mapping an initial distribution of the chain to its distribution at time t. Then point (a) expresses Markovian nature of the chain, and points (b)–(c) are an assumption of continuous dependence on initial data (see Chapter 2 for details).

1.1.2 *An example of a strongly continuous semigroup*

Let r_i, $i \ge 2$ be positive numbers. For any $t \ge 0$, the formula

$$T(t)\,(\xi_i)_{i\ge 1} = (\xi_1 + \sum_{j=2}^{\infty} \xi_j(1-\mathrm{e}^{-r_j t}), \xi_2\mathrm{e}^{-r_2 t}, \xi_3\mathrm{e}^{-r_3 t}, \dots) \tag{1.1}$$

defines a linear operator in $l^1 := l^1(\mathbb{N})$. This operator is bounded with norm not exceeding 1, since

$$\|T(t)x\| \le |\xi_1| + \sum_{j=2}^{\infty} |\xi_j|(1-\mathrm{e}^{-r_j t}) + \sum_{i=2}^{\infty} |\xi_i|\mathrm{e}^{-r_i t} = \sum_{i=1}^{\infty} |\xi_i| = \|x\|.$$

This norm in fact equals 1, because for nonnegative $x \in l^1$ the inequality in the calculation presented above may be replaced by the equality. It is also easy to see, and the reader should check it, that the semigroup property holds.

We claim that $\{T(t), t \ge 0\}$ is a strongly continuous semigroup. Indeed, for any $x \in l^1$,

$$\|T(t)x - x\| \le 2\sum_{j=2}^{\infty} |\xi_j|(1-\mathrm{e}^{-r_j t}),$$

and the right-hand side converges to 0, as $t \to 0+$ by the Lebesgue Dominated Convergence Theorem (see, e.g., Section 1.6.1). Alternatively, for a nonzero x, and given $\epsilon > 0$, one may find $i_0 \in \mathbb{N}$ such that $\sum_{i=i_0+1}^{\infty} |\xi_i| < \frac{\epsilon}{4}$, and then a $t_0 > 0$ such that $\sup_{i=1,\dots,i_0}(1 - \mathrm{e}^{-r_i t}) < \frac{\epsilon}{4\|x\|}$ for all $t \in [0, t_0]$. For such t,

$$2\sum_{j=2}^{\infty} |\xi_j|(1-\mathrm{e}^{-r_j t}) \le 2\sum_{j=2}^{i_0} |\xi_j|(1-\mathrm{e}^{-r_j t}) + 2\sum_{j=i_0+1}^{\infty} |\xi_j|(1-\mathrm{e}^{-r_j t})$$
$$\le \frac{\epsilon}{2\|x\|}\sum_{j=2}^{i_0} |\xi_j| + 2\sum_{j=i_0+1}^{\infty} |\xi_j| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves the claim.

To gain some intuition about what the semigroup defined above describes, consider the following stochastic movement with state-space $\mathbb{N}$: a traveler starting at an $i \ge 2$ stays there for an exponential time with parameter r_i (so that the probability that the traveler is still at i at time t is $\mathrm{e}^{-r_i t}$) and then jumps to the state 1 to stay there for ever. A traveler starting at 1 simply stays there for ever. If ξ_i is the probability that a traveler starts at i, then the ith coordinate of the vector $T(t)\,(\xi_i)_{i\ge 1}$ is the probability that the traveler will be at i at time $t \ge 0$.

1.1.3 *A semigroup that is not strongly continuous*

Let (1.1) be modified so that the first coordinate of the vector on the right-hand side is ξ_1. Then, $T(t)$ is again a linear operator with norm 1 (corresponding to stochastic movement in which a traveler starting at $i \ge 2$ after an exponential time with parameter r_i disappears). The so-modified family $\{T(t), t \ge 0\}$ may be checked to be a strongly continuous semigroup.

The same (modified) formula defines also a semigroup of operators of norm 1 in the space l^∞ of bounded sequences $(\xi_i)_{i\ge 1}$. (Here, the norm is given by $\|x\| = \sup_{i\in\mathbb{I}} |\xi_i|$.) However, if $\sup_{i\ge 2} r_i = \infty$, this semigroup is not strongly continuous. To see this, consider the vector $x = (1, 1, 1, \dots)$. Then, for each $t > 0$,

$$\|T(t)x - x\| = \|(0, 1-\mathrm{e}^{-r_2 t}, 1-\mathrm{e}^{-r_3 t}, \dots)\| = \sup_{i\ge 2} |1-\mathrm{e}^{-r_i t}| = 1.$$

Hence, $T(t)x$ cannot converge to x in the norm of l^∞, as $t \to 0$, showing that $\{T(t), t \ge 0\}$ is not a strongly continuous semigroup in l^∞.

1.1.4 *Continuity*

A closer inspection of Example 1.1.2 shows that the map $[0, \infty) \ni t \mapsto T(t)x$ is continuous in the sense of the norm in l^1 for all $x \in l^1$. This is not a coincidence: it can be shown that this is the property of all C_0 semigroups. In other words, assumptions (a)–(c) of the definition of a strongly continuous semigroup imply that the functions $[0, \infty) \ni t \mapsto T(t)x$, sometimes termed trajectories of the semigroup, are continuous for all $x \in \mathbb{X}$.

1.1.5 *The generator*

Closed form expressions for semigroups of operators are seldom available. But, fortunately, the semigroups can be fully described by their 'derivates,' that is, certain operators, called generators. The situation is somewhat similar to the fact that even if we know initial position $s(0)$ and an exact formula for velocity $v(t)$ of a moving object at any time $t \geq 0$ in terms of elementary functions, a closed form for its position $s(t)$ at time $t \geq 0$ in terms of such functions may be hard to find, or in fact may not exist (take, e.g., $v(t) = \mathrm{e}^{-t^2}$). This is despite the fact that the relation between velocity and position is very well known and simple: $s(t) = s(0) + \int_0^t v(t')\,\mathrm{d}t'$. In terms of Markov chains, the semigroup contains all the information on transition probabilities, whereas its generator gathers the information on transition rates (and more).

Formally, the infinitesimal generator (or simply: the generator) of $(T(t))_{t\geq 0}$ is defined by

$$\boxed{Ax = \lim_{t\to 0+} t^{-1}(T(t)x - x)}$$

for those x for which the limit exists in the sense of the norm in $\mathbb{X}$.

The second part of the previous sentence is important: in situations of interest, the limit presented above rarely exists for all $x \in \mathbb{X}$. If it does, A turns out to be bounded, and the semigroup $\{T(t), t \geq 0\}$ can be recovered from A by means of the following formula:

$$T(t) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \qquad t \geq 0; \tag{1.2}$$

the series here converges in the operator norm. The semigroup generated by a bounded operator A will be denoted $\{\mathrm{e}^{tA}, t \geq 0\}$, and referred to as an exponent of A.

Let's look at some examples; we will come back to our discussion of generators in Section 1.1.11.

1.1.6 *An example of a generator*

Bounded linear operators in $\mathbb{R}^3$ may be identified with 3×3 matrices. In this sense, the formula

$$T(t) = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 5 - 4\mathrm{e}^{-t} - \mathrm{e}^{-6t} & 2\mathrm{e}^{-t} + 3\mathrm{e}^{-6t} & 2\mathrm{e}^{-t} - 2\mathrm{e}^{-6t} \\ 5 - 6\mathrm{e}^{-t} + \mathrm{e}^{-6t} & 3\mathrm{e}^{-t} - 3\mathrm{e}^{-6t} & 3\mathrm{e}^{-t} + 2\mathrm{e}^{-6t} \end{bmatrix}$$

defines a semigroup of operators in $\mathbb{R}^3$; and this is regardless of whether $T(t)x$ is the product of the matrix $T(t)$ and a three dimensional column vector x or the product of a three dimensional row vector x and the matrix $T(t)$. A direct calculation shows that the limit $\lim_{t\to 0+} t^{-1}(T(t)x - x)$ exists and equals Ax, where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 2 \\ 0 & 3 & -3 \end{bmatrix},$$

for all $x \in \mathbb{R}^3$. Therefore, the generator here may be identified with the matrix A.

1.1.7 *An example of application of* (1.2)

Take $a \ge 0$ and $b \ge 0$ such that $a + b > 0$, and let $\mathbb{X}$ be $\mathbb{R}^2$ equipped with any of the equivalent norms. The space $\mathcal{L}(\mathbb{X})$ may be identified with the space of 2×2 matrices. Let's find e^{tA} for $A = \begin{pmatrix} -a, & a \\ b, & -b \end{pmatrix}$, using (1.2). We claim that

$$\mathrm{e}^{tA} = \frac{1}{a+b}\begin{pmatrix} b + a\mathrm{e}^{-(a+b)t}, & a - a\mathrm{e}^{-(a+b)t} \\ b - b\mathrm{e}^{-(a+b)t}, & a + b\mathrm{e}^{-(a+b)t} \end{pmatrix}. \tag{1.3}$$

To prove this, we first introduce

$$B := A + (a+b)I_{\mathbb{X}} = \begin{pmatrix} b, & a \\ b, & a \end{pmatrix}.$$

Since $B^2 = (a+b)B$, we have, by induction, $B^n = (a+b)^{n-1}B$. It follows that

$$\mathrm{e}^{tB} = I_{\mathbb{X}} + \frac{1}{a+b}\sum_{n=1}^{\infty} \frac{t^n (a+b)^n}{n!} B = I_{\mathbb{X}} + \frac{1}{a+b}(\mathrm{e}^{(a+b)t} - 1)B.$$

Next, we note that, similarly as for complex numbers, $\mathrm{e}^{t(B+C)} = \mathrm{e}^{tB}\mathrm{e}^{tC}$ provided $BC = CB$. Since $I_{\mathbb{X}}$ commutes with any operator (matrix), by the definition of B, we obtain

$$\begin{aligned} e^{tA} &= e^{-(a+b)t} e^{tB} = e^{-(a+b)t} I_{\mathbb{X}} + \frac{1}{a+b}(1 - e^{-(a+b)t})B \\ &= \frac{1}{a+b}(B - e^{-(a+b)t} A), \end{aligned}$$

completing the proof.

1.1.8 *Another example of application of* (1.2)

Let $\mathbb{X} = C[0, \infty]$ be the space of continuous functions on $[0, \infty)$ with limits at ∞. Equipped with the supremum norm, $\mathbb{X}$ is a Banach space, and for any $a > 0$, the operator defined by $Ax(p) = a[x(p+1) - x(p)]$, $p \geq 0$ is bounded, since $\|Ax\| \leq 2a\|x\|$. To compute e^{tA} for $t \geq 0$, we use the definition and the fact that $B := A + aI_{\mathbb{X}}$ is a scalar multiple of the shift operator: $Bx(p) = ax(p+1)$, $p \geq 0$ so that $B^n x(p) = a^n x(p+n)$, $p \geq 0$. Therefore,

$$\begin{aligned} e^{tA} x(p) &= e^{-at} e^{t(A + aI_{\mathbb{X}})} x(p) = \sum_{n=0}^{\infty} e^{-at} \frac{a^n t^n}{n!} x(p+n) \\ &= \mathbb{E}\, x(p + N(t)), \qquad t \geq 0, \end{aligned}$$

where $\mathbb{E}$ denotes expected value. In the last line $N(t)$ is a Poisson-distributed random variable with parameter at:

$$\mathbb{P}(N(t) = k) = e^{-at} \frac{(at)^k}{k!}.$$

In other words, the exponent $\{e^{tA}, t \geq 0\}$ describes a Poisson process. In this process, if the starting point is p then after time $t \geq 0$ with probability $\mathbb{P}(N(t) = k)$ the process is at $p + k$.

1.1.9 *Example: The generator of the semigroup of Section 1.1.2*

Let us come back to Example 1.1.2, and, to focus our attention, let's agree that $r_i = i$, $i \geq 2$. To find the generator of the semigroup (1.1), we note first that convergence in the sense of l^1 norm implies convergence of all coordinates. Therefore, if $x = (\xi_i)_{i \geq 1}$ is in the domain $\mathcal{D}(A)$ of the generator, then the ith coordinate of Ax must be $\lim_{t \to 0+} \frac{e^{-it} - 1}{t} \xi_i = -i\xi_i$, $i \geq 2$. Since Ax, being the limit of elements of l^1, belongs to l^1, we see that a necessary condition for x to belong to $\mathcal{D}(A)$ is that

$$\sum_{i=2}^{\infty} i|\xi_i| < \infty. \tag{1.4}$$

We will show that all x's satisfying this condition belong to $\mathcal{D}(A)$ and that for such x, Ax is equal to

$$y := (\sum_{i=2}^{\infty} i\xi_i, -2\xi_2, -3\xi_3, \cdots).$$

For, under assumption (1.4), y belongs to l^1. Moreover, the ith coordinate of $t^{-1}(T(t)x - x) - y$ is

$$\begin{cases} \left[\frac{e^{-it}-1}{t} + i\right]\xi_i, & i \geq 2, \\ -\sum_{j=2}^{\infty} \xi_j \left[\frac{e^{-jt}-1}{t} + j\right], & i = 1, \end{cases}$$

and we have $\frac{e^{-it}-1}{t} + i = \frac{i}{t}\int_0^t (1 - e^{-is})\,ds$. Therefore,

$$\|t^{-1}(T(t)x - x) - y\| \leq 2\sum_{i=2}^{\infty} i|\xi_i|\frac{1}{t}\int_0^t (1 - e^{-is})\,ds.$$

Since $\lim_{t\to 0+} \frac{1}{t}\int_0^t (1 - e^{-is})\,ds = 0$ and $0 \leq \frac{1}{t}\int_0^t (1 - e^{-is})\,ds \leq 1$, $i \geq 2$, the right-hand side above converges to 0 by assumption (1.4) and the Lebesgue Dominated Convergence Theorem. (Alternatively, one may argue as in Example 1.1.2.) This shows that $x \in \mathcal{D}(A)$ and $Ax = y$.

1.1.10 *Example: Isomorphic semigroups*

Suppose that Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic: there is a bounded linear map $I\colon \mathbb{Y} \to \mathbb{X}$ with bounded left and right inverse $I^{-1}\colon \mathbb{X} \to \mathbb{Y}$ so that $II^{-1} = I_{\mathbb{X}}$ and $I^{-1}I = I_{\mathbb{Y}}$. Suppose also that $\{T(t), t \geq 0\}$ is a strongly continuous semigroup in $\mathbb{X}$ with generator A. Then the operators

$$I^{-1}T(t)I, \quad t \geq 0$$

form a strongly continuous semigroup in $\mathbb{Y}$. Since the limit

$$\lim_{t\to 0+} \frac{I^{-1}T(t)Iy - y}{t}$$

exists iff so does

$$\lim_{t\to 0+} \frac{T(t)Iy - Iy}{t},$$

a y belongs to the domain of the infinitesimal generator, say, $A^{\diamond}$, of $\left\{I^{-1}T(t)I, t \geq 0\right\}$ iff Iy belongs to $\mathcal{D}(A)$, and then

$$A^{\diamond}y = I^{-1}AIy.$$

The semigroups $\left\{I^{-1}T(t)I, t \geq 0\right\}$ and $\{T(t), t \geq 0\}$, which of course play symmetric roles, are said to be **isomorphic** to each other, and so are said to be the generators A and $A^{\diamond}$; other authors prefer to speak of **similar semigroups**. Sometimes, one of the isomorphic semigroups/generators is easier to handle and knowing the isomorphism involved aids the analysis of the other. See, for example, Exercise 1.1.18 or Section 3.3.7.

1.1.11 *Generators are densely defined*

As already stated, the limit defining the generator rarely exists for all $x \in \mathbb{X}$; $\mathcal{D}(A)$ is rarely equal to the entire $\mathbb{X}$. In fact, $\mathcal{D}(A) = \mathbb{X}$ iff the generator is bounded, and this happens iff

$$\lim_{t \to 0+} \|T(t) - I_{\mathbb{X}}\| = 0,$$

which is a much stronger condition than assumption (c) of the definition of Section 1.1.1.

All we can say at the first inspection of $\mathcal{D}(A)$ is that it is a linear subset of $\mathbb{X}$. Fortunately, $\mathcal{D}(A)$ turns out to be always dense in $\mathbb{X}$: any $x \in \mathbb{X}$ may be approximated by elements of $\mathcal{D}(A)$. In particular, for generators of semigroups we need to search in the class of densely defined operators.

The reason for $\mathcal{D}(A)$ to be dense is as follows. Since the trajectories $t \mapsto T(t)x$ are continuous functions of t, one may think of Riemann-type integrals $\int_a^b T(t)x \,\mathrm{d}t$ for any $0 \leq a < b$. These are defined in the same way as integrals of real-valued functions, as limits of approximating Riemann sums, and possess similar properties. For example, continuity of a function $f\colon [a,b] \to \mathbb{X}$ guarantees that the integral $\int_a^b f(t)\,\mathrm{d}t$ is well defined, provided $\mathbb{X}$ is a Banach space (see, e.g., [14], pp. 60–63), and

$$\left\| \int_a^b f(t)\,\mathrm{d}t \right\| \leq \int_a^b \|f(t)\|\,\mathrm{d}t. \tag{1.5}$$

(This estimate will be found useful later.) Therefore, one may think of $\int_0^h T(t)x\,\mathrm{d}t$ for any $h > 0$ and $x \in \mathbb{X}$, and it can be seen that such elements belong to $\mathcal{D}(A)$ (with $A\int_0^h T(t)x\,\mathrm{d}t = T(t)x - x$). Moreover, it is a well-known property of Riemann integrals that for a continuous function f on an interval $[a,b]$,

$$\lim_{h \to 0+} h^{-1} \int_s^{s+h} f(t)\,\mathrm{d}t = f(s), \qquad s \in [a,b),$$

and Riemann integrals of vector-valued functions also possess this property. Applying this to $f(t) = T(t)x$ and $s = 0$, we obtain

$$\lim_{h \to 0+} h^{-1} \int_0^h T(t)x \, dt = x.$$

This however means that any $x \in \mathbb{X}$ may be approximated by elements of $\mathcal{D}(A)$.

1.1.12 *Generators are closed*

Generators of semigroups, though generally unbounded, are closed. By definition this means that conditions

$$x_n \in \mathcal{D}(A), n \geq 1, \qquad \lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} Ax_n = y$$

imply that x belongs to $\mathcal{D}(A)$ and $Ax = y$. In other words, the graph of A, defined as the following subset of the Banach space $\mathbb{X} \times \mathbb{X}$:

$$\mathbb{G}_A = \{(x, y) \in \mathbb{X} \times \mathbb{X}; x \in \mathcal{D}(A), y = Ax\}$$

is closed in $\mathbb{X} \times \mathbb{X}$.

For example, it may be checked directly that the generator we calculated in Section 1.1.9 is closed. That the latter operator is not bounded (in the sense that there is no M such that $\|Ax\| \leq M\|x\|$ for $x \in \mathcal{D}(A)$) may be seen by considering the vectors $e_i \in l^1$ which at the ith coordinate have 1 and are composed of 0's otherwise: since $Ae_i = -ie_i$, which implies $\|Ae_i\| = i\|e_i\|$, the hypothetical constant M would need to be larger than all $i \in \mathbb{N}$.

Even though the notion of closedness is very important, in searching for generators of semigroups one rarely needs to prove directly that a candidate for a generator is closed. This is because, even if A is already proved to be densely defined and closed, before one can claim that A is a generator, another condition, discussed in the next section (Section 1.2), needs to be checked, and operators satisfying this condition are automatically closed.

1.1.13 *Closability*

Analysis of generators of Markov chains often involves closable operators. To recall, a linear operator A in a Banach space $\mathbb{X}$ is said to be closable if conditions

$$x_n \in \mathcal{D}(A), n \geq 1 \qquad \lim_{n \to \infty} x_n = 0 \qquad \text{and} \qquad \lim_{n \to \infty} Ax_n = y$$

imply $y = 0$. When combined with linearity this condition means that the closure of the graph of A in the norm of $\mathbb{X} \times \mathbb{X}$ is still a graph (of another operator). The latter, 'larger' operator, termed the closure of A and denoted $\overline{A}$, is defined by

$$\overline{A}x = \lim_{n\to\infty} Ax_n$$

on the domain composed of $x \in \mathbb{X}$ for which there are $x_n \in \mathcal{D}(A)$ such that $\lim_{n\to\infty} x_n = x$ and the limit $\lim_{n\to\infty} Ax_n$ exists; the definition of $\overline{A}x$ does not depend on the choice of $(x_n)_{n\ge 1}$ precisely because A is closable (cf. point 2 of the proof presented in Section 5.4.7).

For example, suppose that there is a closed operator B such that

$$\mathbb{G}_A \subset \mathbb{G}_B.$$

Then A is automatically closable, and $\overline{A}$ is simply the restriction of B to $\mathcal{D}(\overline{A})$. In fact, $\overline{A}$ (if it exists) is the smallest (in the sense of inclusion of graphs) closed operator extending A.

1.1.14 *Cores*

Sometimes one faces a situation that is in a sense 'inverse' to the one discussed above: a closed operator A and a subset $\mathcal{D}$ of its domain are given, and the question is whether for each $x \in \mathcal{D}(A)$ there is a sequence $(x_n)_{n\ge 1}$ of elements of $\mathcal{D}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = Ax$. If this is the case, $\mathcal{D}$ is said to be a core for A. Since the graph of $A_{|\mathcal{D}}$ (of the operator A restricted to $\mathcal{D}$) is a subset of $\mathbb{G}_A$, closability of $A_{|\mathcal{D}}$ is not an issue here: instead, we would like to know whether $\mathcal{D}$ is 'large enough' so that A may be 'recovered' from $A_{|\mathcal{D}}$. Such information is vital in the situations in which it is hard to describe the entire $\mathcal{D}(A)$ but a manageable description of $\mathcal{D}$ is available. As an easy exercise, the reader may check to see that the set of linear combinations of basis vectors is a core for the generator A of Section 1.1.9.

Before closing this section, we need to say a word about connection between semigroups and Cauchy problems.

1.1.15 *Semigroups and Cauchy problems*

As commented in 1.1.4, the assumption of continuity of $t \mapsto T(t)x$, at $t = 0$ for all $x \in \mathbb{X}$ implies, via the semigroup property, that $t \mapsto T(t)x$ is continuous at all $t \ge 0$. Similarly, if x belongs to $\mathcal{D}(A)$ then, by definition, $t \mapsto T(t)x$ is differentiable at $t = 0$ with (right-hand) derivative equal Ax. The semigroup property allows extending this attribute of $t \mapsto T(t)x$ to the entire half-line: it can be shown that this map is continuously differentiable there, that $T(t)x$ belongs to $\mathcal{D}(A)$ for all $t \ge 0$, and that

$$\frac{\mathrm{d}}{\mathrm{d}t}(T(t)x) = AT(t)x = T(t)Ax, \qquad t \ge 0.$$