VISCOELASTIC WAVES AND RAYS IN LAYERED MEDIA

This second edition extends the rigorous, self-contained exposition of the theory for viscoelastic wave propagation in layered media to include head waves and general ray theory. The theory, not published elsewhere, provides solutions for fundamental wave-propagation and ray-theory problems valid for any media with a linear response, elastic or anelastic. It explains measurable variations in wave speed, particle motion, and attenuation of body waves, surface waves, and head waves induced at anelastic material boundaries that do not occur for elastic waves.

This book may be used as a textbook for advanced university courses and as a research reference in seismology, exploration geophysics, engineering, solid mechanics, and acoustics. It provides computation steps for ray-tracing computer algorithms to develop a variety of tomography-inferred anelastic models, such as those for the Earth's deep interior and petroleum reserves. Numerical results and problem sets emphasize important aspects of the theory for each chapter.

Roger Borcherdt is a research seismologist, Scientist Emeritus at the U.S. Geological Survey, and past consulting and visiting Shimizu Professor at Stanford University. Dr. Borcherdt is the author of more than 200 scientific publications, including several on the theoretical and empirical aspects of seismic wave propagation pertaining to problems in seismology, exploration geophysics, and earthquake engineering. He is the recipient of the Distinguished Service Award, the highest honor of the Department of Interior, for seminal contributions in seismology and engineering; the Bruce A. Bolt Medal, awarded jointly by the Seismological Society of America, the Earthquake Engineering Research Institute, and COSMOS; and the 1994 and 2002 Outstanding Paper Awards of *Earthquake Spectra*. He is an Honorary Member of the Earthquake Engineering Research Institute, a past journal editor, and a co-inventor of the General Earthquake Observation System (GEOS, patent 4,603,486), as well as an active member of several professional societies.

Reviews of the 1st edition

"This is a fascinating book that deals with the theory and application of wave propagation in lossy materials. It is highly recommended reading for anyone who deals with seismic or acoustic waves in real media."

Journal of the Acoustical Society of America

"This book is a welcome and needed addition to the literature of linear wave propagation in layered media. It would be perfect as a complementary text in a wave propagation course, as it fills a void that has existed until now in current texts."

Earthquake Spectra

Praise for the 2nd edition

"This is a significant advancement to the theory of viscoelasticity. This book is a tour de force and could be the vanguard for a whole branch of seismology focused on the importance of lossy media. This would be an excellent textbook for a graduate course in theoretical seismology."

Ralph A. Stephen, Woods Hole Oceanographic Institution

"The chapter on *General Viscoelastic Ray Theory* is a significant and welcome addition to the first edition. This textbook is also a research treatise that should gain in importance over the next decade with the need to account for viscoelastic material characteristics evident in improved data sets."

Michael A. Slawinski, Memorial University of Newfoundland, Canada

Cover illustration: Amplitude response of a viscoelastic layer to an incident inhomogeneous Type-II S wave and distinct ray paths for body waves in elastic and anelastic media induced by gradients in intrinsic material absorption (Figures (9.3.13c) and (11.7.1)).

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In Memory of My Mother and Father
Dedicated to my Family

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Preface

The first edition of this book provides a self-contained mathematical exposition of the theory of monochromatic wave propagation in layered viscoelastic media. This second edition extends the exposition to include the theory of general viscoelastic rays and head waves as derived from first principles. It provides the general solutions for many of the fundamental time-harmonic and ray-theory problems of seismology for layered anelastic media, not previously published in another book. As a text book with numerical examples, problem sets and appended solutions, this book provides the opportunity to teach the theory of monochromatic wave propagation and corresponding ray theory as usually taught for elastic media in the broader context of waves and rays in any media with a linear response without undue complications in the mathematics. The general formulations used for the constitutive relation and the wave solutions afford considerable generality and simplification in the mathematics required to derive analytic solutions valid for any viscoelastic medium including the special case of an elastic medium. The book is intended for advanced undergraduate and graduate students of wave propagation with prerequisites being knowledge of differential equations and complex variables.

As a reference text, this book provides the theory of viscoelastic monochromatic wave propagation in more than one dimension and corresponding ray theory developed in the last five decades. As such, the book provides a compendium of recent advances that describe the physical characteristics of two and three dimensional anelastic body and surface waves that are not predictable based on the theory for one-dimensional waves. It predicts the existence of general anelastic rays with associated travel-time and amplitude attenuation characteristics that do not exist in elastic media. It provides the basis for the derivation of results beyond the scope of the present textbook. The theory is of interest in the broad field of solid mechanics and of special interest in seismology, engineering, exploration geophysics, and acoustics for consideration of wave propagation and ray theory in layered media with arbitrary amounts of intrinsic absorption, ranging from low-loss models of the deep Earth to moderate loss models for soils and weathered rock.

The phenomenological constitutive theory of linear viscoelasticity dates to the nineteenth century with the application of the superposition principle in 1874 by Boltzmann. He proposed a general mathematical formulation that characterizes linear anelastic as well as elastic material behaviour. Subsequent developments in the theory did not occur until interest increased in the response of polymers and other synthetic materials during the middle of the twentieth century. During this time considerable simplification of the mathematical aspects of the theory occurred with developments in the theory of linear functionals (Volterra 1860-1940) and the introduction of integral transform techniques. Definitive accounts and contributions to the mathematical theory of viscoelasticity include the works of Volterra (2005), Gross (1953), Bland (1960), Fung (1965), Flügge (1967), and the rigorous account by Gurtin and Sternberg (1962). Gross (1953) suggested that “The theory of viscoelasticity is approaching completion. Further progress is likely to be made in applications rather than on fundamental principles." Hunter in 1960 indicated that the application of the general theory of linear viscoelasticity to other than one dimensional wave-propagation problems was incomplete. Subsequent to Hunter’s review, significant advances in the application of viscoelasticity to wave propagation in layered two- and three-dimensional media occurred in the early 1970s and have continued with the most recent being the development of general viscoelastic ray theory. The exposition presented herein provides a self-contained mathematical account of these developments. A brief historical summary of major developments and corresponding discoveries are provided in the next section entitled “Historical Prologue”.

Recent theoretical developments show that the physical characteristics of the predominant types of body waves that propagate in layered anelastic media are distinctly different from the predominant type in elastic media or one-dimensional anelastic media. For example, two types of shear waves propagate each with different amounts of attenuation. Physical characteristics of anelastic waves refracted across anelastic boundaries, such as particle motion, phase and energy velocities, energy loss, and direction of energy flux vary with angle of incidence. Hence, the physical characteristics of the waves propagating through a stack of layers are no longer unique in each layer as they are in elastic media, but instead depend on the angle at which the wave entered the stack. These fundamental differences explain laboratory observations of reflected amplitudes, head wave arrivals, observed travel times and amplitude attenuation not explained by elasticity or one dimensional viscoelastic wave-propagation theory. These differences lend justification to the need for a book with a rigorous mathematical treatment of the fundamental aspects of viscoelastic wave propagation and ray theory in layered media.

Viscoelastic material behaviour is characterized herein using Boltzmann’s formulation of the constitutive law. Analytic solutions derived for various problems are valid for both elastic and linear anelastic media as might be modelled by an infinite number of possible configurations of elastic springs and viscous dashpots. The general analytic solutions are valid for viscoelastic media with an arbitrary amount of intrinsic absorption. The theory, based on the assumption of linear strain, is valid for media with a nonlinear response to the extent that linear approximations are valid for sufficiently small increments in time. Wave propagation and ray theory results in this book are based on those of the author as previously published and explicitly derived herein including recent results for multilayered media, Love-Type surface waves, viscoelastic ray theory, and viscoelastic head waves.

The book is intended for use in a graduate or upper division course and as a reference text for those interested in wave propagation and general ray theory in layered media. The book provides theoretical solutions for many of the fundamental monochromatic wave-propagation problems of seismology in the context of a self-contained treatment as derived from first principles valid for any viscoelastic media, elastic or anelastic. The book provides solutions and numerical results for problems concerned with the propagation, reflection, and refraction of P, SI, and SII body waves, response of a stack of multiple layers, Love and Rayleigh type surface waves, head waves, and general viscoelastic ray theory. The book offers students the opportunity to understand classic elastic results for these problems in the broader context of wave propagation in any material with a linear response.

Chapter 1 provides an introduction for new students to basic concepts of a linear stress-strain law, energy dissipation, and wave propagation for one-dimensional linear viscoelastic media. It provides examples of specific models derivable from various configurations of springs and dashpots as special cases of the general formulation.

Chapter 2 extends the basic concepts for viscoelastic media to three dimensions. It provides the general linear stress-strain law, notation for components of stress and strain, the equation of motion, and a rigorous account of energy balance as the basis needed for a self-contained treatment of viscoelastic wave propagation in more than one dimension.

Chapter 3 provides a thorough account of the physical characteristics of harmonic waves in three-dimensional viscoelastic media. It provides closed-form expressions and corresponding quantitative estimates for characteristics of general (homogeneous or inhomogeneous) P, Type-I, and Type-II S waves in viscoelastic media with both arbitrary and small amounts of intrinsic absorption. It includes expressions for the physical displacement and volumetric strain associated with various wave types. This chapter is a prerequisite for analytic solutions derived for reflection-refraction and surface-wave problems in subsequent chapters.

Chapter 4 specifies the expressions for monochromatic P, Type-I S, and Type-II S wave solutions needed in subsequent chapters to solve reflection-refraction and surface-wave problems in layered viscoelastic media. The solutions are expressed in terms of the propagation and attenuation vectors and in terms of the components of the complex wave numbers.

Chapter 5 provides analytic closed-form solutions for the problems of general (homogeneous or inhomogeneous) P, Type-I S, and Type-II S waves incident on a plane boundary between viscoelastic media. Conditions for homogeneity and inhomogeneity of the reflected and refracted waves and the characteristics of energy flow at the boundary are derived. Careful study of results for the problems of an incident Type-I S and Type-II S wave is useful for understanding reflection-refraction phenomena and energy flow due to an inhomogeneous wave incident on a viscoelastic boundary.

Chapter 6 provides numerical results for various single boundary reflection-refraction problems using the analytic solutions derived in the previous chapter. Examples are chosen to provide quantitative estimates of the physical characteristics for materials with moderate and small amounts of intrinsic material absorption as well as a comparison with laboratory measurements in support of theoretical predictions. Study of these examples, especially the first three, is recommended for developing an improved understanding of the physical characteristics of waves reflected and refracted at viscoelastic boundaries.

Chapter 7 provides theoretical solutions and quantitative results for problems of a general Type-I S, P, or Type-II S wave incident on the free surface of a viscoelastic half space. Results are included to facilitate understanding and interpretation of measurements as might be detected on seismometers and volumetric strain meters at or near the free surface of a viscoelastic half space.

Chapter 8 presents the analytic solution and corresponding numerical results for a Rayleigh-Type surface wave on a viscoelastic half space. The anelastic viscoelastic solution reveals various characteristics including those for particle motion, vertical amplitude distribution, and attenuation that are not predicted by the elastic solution as originally derived by Lord Rayleigh.

Chapter 9 provides the analytic solution for the response of multilayered viscoelastic media to an incident general Type-II S wave. It provides numerical results for elastic, low-loss, and moderate-loss viscoelastic media useful in understanding variations in response of a single layer due to inhomogeneity and angle of incidence of the incident wave.

Chapter 10 provides the analytic solution and corresponding numerical results for a Love-Type surface wave on multilayered viscoelastic media. It derives roots of the resultant complex period equation for a single layer needed to provide curves showing the dependencies of wave speed, attenuation coefficient, and amplitude distribution on frequency, intrinsic absorption, and other material parameters for the fundamental and first higher mode.

New students desiring a basic understanding of Rayleigh-Type surface waves, the response of a stack of viscoelastic layers to incident inhomogeneous waves, and Love-Type surface waves will benefit from a thorough reading of Chapters 8, 9, and 10.

Chapter 11 provides the analytic solutions and numerical results for general viscoelastic rays and head waves in horizontal and spherical layered media with intrinsic material gradients. It provides the solutions for the forward and inverse ray tracing problems. It defines viscoelastic ray parameters for phase propagation and amplitude attenuation that account for variations in inhomogeneity of the waves along the ray path and corresponding variations in travel time and amplitude attenuation. It provides computation steps for computer codes to trace general viscoelastic rays which in turn specify earth-flattening transformations for media with layers and gradients. It provides solutions of inverse problems to infer intrinsic material constitutive properties from empirical measurements of travel time and amplitude. It extends the elastic solutions of the Herglotz-Wiechert integral to viscoelastic horizontal and spherical media. It discusses possible shortcomings of using elastic models to explain observations of travel time and amplitude in layered anelastic materials, such as those in the Earth. It discusses why development of two- and three- dimensional ray tracing algorithms based on general viscoelastic theory will predict travel-time and amplitude attenuation curves distinct from those predicted by elastic models or elastic models with superimposed one-dimensional attenuation.

Students interested in developing an understanding of the fundamentals of general ray theory will benefit from a thorough reading of Section 11.1 for horizontal layers with special attention to the analytic and numerical examples. Students of the anelastic internal structure of the Earth will find additional benefit in the sections pertaining to material gradients and spherical layers (11.2-11.4) and to the discussion of implications of using elastic models to account for empirical observations in anelastic media (11.6.3). Students interested in developing computer codes for tracing general viscoelastic rays in layered anelastic media, such as the Earth, will be interested in Section 11.5. Those concerned with the inverse problem of inference of intrinsic material parameters from empirical measurements will be interested in Section 11.6.

Chapter 12 provides appendices that augment material presented in preceding chapters. They include integral identities, vector identities, and derivations relegated to the appendices to facilitate readability of the main text.

For students interested in history, the *Historical Prologue* provides a brief summary of the mathematical solutions developed throughout the book for fundamental anelastic wave propagation and ray theory problems. Corresponding discoveries of new characteristics of anelastic waves and rays implied by the anelastic solutions also are indicated.

A special note of respect is due those that have developed the elegant constitutive theory of linear viscoelasticity, such as Boltzmann, Volterra, Gurtin, and Sternberg. Their important contributions make applications such as that presented here possible. I would like to acknowledge the initial guidance of the late Professor Jerome L. Sackman, whose expertise in viscoelasticity started a theoretical journey I could not have imagined. The excellent insightful reviews of the first edition in Earthquake Spectra (26, 2010) by Professor J. Bielak and in the Journal of the Acoustical Society of America (126, 2009) by Professor R. A. Stephen, and in Mathematical Reviews have contributed to the book's success.

A special note of acknowledgement is due Professor Ivan Pšenčík, whose stimulating scientific discussion at a workshop in 2015, inspired development of viscoelastic ray theory from first principles as presented in Chapter 11. The mathematically insightful and broad-reaching review of the 2nd edition for Cambridge University Press (CUP) by Professor R. A. Stephen was of special benefit. CUP review comments received from Professors M. Slawinski and E. Krebes are appreciated as are comments on advance copies provided to Professors J. Bielak, K. Innanen, P. Marston, I. Pšenčík, and K. Law on November 8-12, 2017, to Professor E. Safak and Dr. S. Moradi on May 23, 2018 and to Professor J. Anderson on June 29, 2018. Review comments on advance copies of the 1st edition by Professor J. Bielak, Dr. W. H. K. Lee, Professor C. Langston, and the late Professor J. Sackman are appreciated as are contributions to previously published papers by co-authors G. Glassmoyer and L. Wennerberg.

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A special note of appreciation is due my family (Darren, Ryan, and Deborah) for their patience and support for the many late hours needed to finish the first and second editions. Without their understanding the opportunity to experience the personal excitement associated with discovering characteristics of waves and rays basic to seismology and engineering through the elegance and rigors of mathematics would not have been possible.

Roger D. Borcherdt

Historical Prologue

This book represents an unimagined mathematical journey that has led to the general solution of many of the fundamental monochromatic problems of seismology for layered viscoelastic media. These general solutions reveal several characteristics of seismic and acoustic waves not previously apparent from elastic solutions.

Historically, this exciting journey began in 1969 with an introductory meeting requested by the author to seek the advice of the late Professor Jerome Sackman, whose expertise in the viscoelastic response of structural materials was well known. To infer the damping characteristics of soils using empirical observations of Rayleigh waves, the author as a graduate student derived a solution of the complex Rayleigh equation. His solution implied that the propagation and attenuation vectors of the component solutions were not parallel and normal to the free surface, while those for the well-known elastic solution are. Professor Sackman pondered the result for a moment, retrieved a paper from his file, then said “Have you seen this paper by Lockett (1962), regarding incident homogenous waves refracted at an anelastic boundary. He too, has found some unusual characteristics for wave solutions in anelastic media”. Professor Sackman’s insight and the consistency of the preliminary results led to the author’s initial application of the vector Helmholtz equation to the solution of fundamental wave-propagation problems in seismology, to the development of general solutions for incident inhomogeneous anelastic body and surface waves in his PhD dissertation, and to the start of a fascinating journey that only the rigors of mathematics could reveal.

This mathematical journey is marked with historical waypoints corresponding to major developments and corresponding discoveries of characteristics of anelastic seismic and acoustic waves and rays, not previously apparent from elastic solutions. These waypoints together with cross references to equations, figures, text, and the author publication dates as referenced herein are briefly summarized as follows:

(I) Application of the vector Helmholtz equation to describe viscoelastic wave propagation in terms of propagation and attenuation vectors and its resulting solutions ((3.1.3), (3.1.4); 1971, 1973a) that revealed:

(1) The only types of inhomogeneous body and surface waves that propagate in anelastic viscoelastic media do not propagate in elastic

viscoelastic media and vice-versa (Theorems (3.1.17) –(3.1.19); 1971, 1973a),

(2) Physical characteristics of anelastic inhomogeneous P and S body waves, such as phase and energy speeds, elliptical particle motions, mean energy flux, mean energy densities, and fractional energy loss proportional to reciprocal quality factor Q^{-1} that are not unique as they are in elastic media, but dependent on the degree of inhomogeneity of the wave with wave speeds and fractional energy loss less and greater than respectively, those for homogenous waves (Sections 3.1-3.4; 1971, 1973a),

(3) Two types of anelastic inhomogeneous S waves exist (SI and SII) with elliptical and linear particle motions, distinct energy wave velocities, distinct mean potential energy densities, distinct total energy densities, distinct rates of energy dissipation, and distinct reciprocal quality factors $(Q_{SI}^{-1}, Q_{SII}^{-1})$ (Sections 3.1-3.4; 1977, 1982),

(II) CHARACTERIZATION OF STEADY-STATE VISCOELASTIC MATERIAL BEHAVIOR IN TERMS OF WAVE SPEEDS (v_{HS}, v_{HP}) AND INTRINSIC MATERIAL ABSORPTIONS $(Q_{HS}^{-1}, Q_{HP}^{-1})$ AND DENSITY ((3.5.8)-(3.5.11); 1971, 1985) THAT REVEALED:

(4) Physical insight and formula simplification for specification of physical characteristics of viscoelastic P, SI, and SII body waves, reflected and refracted waves, Rayleigh- and Love-Type surface waves, response of multiple layers, and general viscoelastic rays (Sections 3.6, 3.7, Chapters, 5, 7, 8, 9, 10, 11; 1971, 1973a, b, 1977, 1982, 1985, 2017),

(5) Formulas for quantification of the influence of inhomogeneity and intrinsic material absorption on physical characteristics of viscoelastic body and surface waves, with explicit expressions for low-loss media $(Q_{HS}^{-1} << 1)$ (Sections 3.6 and 3.7; 1971, 1985) and relative magnitude of reciprocal quality factors for inhomogeneous SI, SII, and P waves, ((3.6.89), (3.6.92); 2009),

(III) SOLUTIONS FOR VISCOELASTIC REFLECTED AND REFRACTED WAVES GENERATED BY INCIDENT GENERAL (HOMOGENEOUS OR INHOMOGENEOUS) P, SI, OR SII WAVES IN LAYERED MEDIA (CHAPTERS 4, 5, 6, 7; 1971, 1977, 1982) THAT REVEALED:

(6) Generalized Snell's Law as implied by application of welded boundary conditions which shows that components of the propagation and attenuation vectors parallel to the boundary for P, SI, and SII solutions on each side of the boundary are equal to those of the

incident wave, (Theorems (5.2.22), (5.3.16), (5.4.15); 1971, 1977, 1982),

(7) P, SI, and SII body waves refracted across anelastic boundaries with contrasts in intrinsic absorption are inhomogeneous, unless normally incident, with the degree of inhomogeneity and corresponding physical characteristics such as wave speed and attenuation, dependent on angle of incidence (Theorems (5.2.34), (5.2.43), (5.2.44), (5.3.23), (5.3.24), (5.4.20), (5.4.21); 1971,1977, 1982),

(8) Conservation of energy at an anelastic welded plane boundary requires that the normal component of energy flux be continuous with the total incident energy being the sum of the mean perpendicular components carried away from the boundary by the reflected and transmitted waves and the energy flow due to interaction of the velocity and stress fields of the incident and reflected waves and that of the transmitted waves that does not occur in elastic media ((5.4.47)-(5.4.50), Theorems (5.4.27), (5.4.31); 1977, 1982),

(9) Confirmation of viscoelastic plane-wave theory with observed reflection coefficients for water, stainless-steel anelastic boundary (Becker and Richardson, 1969) as explained by energy carried away from the boundary by general anelastic transmitted waves and energy flow due to velocity and stress field interaction, (Sections 6.2.2, 6.2.3; 1986),

(10) Volumetric strain and displacement for P waves in layered anelastic media dependent on degree of inhomogeneity that permits interpretation of simultaneous collocated measurements of strain and displacements to infer characteristics of anelastic body waves not permitted by either measurement alone in solids such as those in the Earth (Sections 3.10, 7.2.2; 1988, 1989a, 1989b),

(11) Measurable variations in anelastic reflection coefficients for Earth free-surface displacements, ocean, solid-Earth boundaries, and near-surface volumetric strains associated with inhomogeneous anelastic body waves that do not occur for elastic waves (Figures (6.1.5)b, (6.110)b, (6.2.8)a, (6.2.10)a, (6.2.11)a, (7.2.49), (7.3.35), (7.3.27); 1971, 1982, 1985, 1986, 1988, 1989a, 1989b),

(IV) SOLUTION FOR RAYLEIGH-TYPE SURFACE WAVE ON A GENERAL VISCOELASTIC HALF SPACE (CHAPTER 8: 1971, 1973B) THAT REVEALED:

(12) For anelastic media directions of phase propagation and attenuation for component anelastic P and SI solutions that are not parallel and perpendicular to the free surface as they are for elastic media resulting in distinct physical characteristics, including; (a) retrograde and prograde elliptical particle motions with tilt of the major axis dependent on depth, and (b) amplitude and volumetric-strain

distributions that exhibit superimposed sinusoidal dependencies on depth that do not occur for elastic media (Sections 8.1, 8.2, 8.3; 1971, 1973b, 1988, 1989a, 1989b),

(13) General sets of numerical solution curves for wave speed, attenuation coefficient, particle motion, and amplitude of Rayleigh-type surface waves versus depth and intrinsic material absorption valid for general viscoelastic media (Figures (8.3.16), (8.3.17), (8.3.19) through (8.3.32); 1971, 1973b, 1988, 1989a),

(V) SOLUTION FOR RESPONSE OF MULTIPLE VISCOELASTIC LAYERS TO INCIDENT GENERAL WAVES (CHAPTER 9; 2009) THAT REVEALED:

(14) Amplitude and phase response of a stack of parallel anelastic viscoelastic layers depends on the angle of incidence, the inhomogeneity of the incident wave, and the intrinsic material absorption in each layer, which are dependencies that do not exist for the response of elastic viscoelastic layers as initially derived by Thompson (1950) and revised by Haskell 1953 (Section 9.1; 2009),

(15) Amplitude response of a surface layer increases for each mode near grazing incidence depending on the amount of inhomogeneity of the incident wave and the amount of intrinsic material absorption (Figure (9.1.13); 2009),

(16) Measurable variations in the response of models of Earth's crustal layers, rock layers, and soft-soil layers due to anelastic material absorption that are not predicted by elastic models (Figures 9.1.11), (9.1.13); 2009),

(VI) GENERAL ANALYTIC SOLUTION FOR LOVE-TYPE SURFACE WAVES IN A HORIZONTAL STACK OF MULTILAYERED VISCOELASTIC MEDIA (SECTIONS 10.1, 10.2, 10.3; 2009) THAT REVEALED:

(17) General solutions for the wave speed and attenuation coefficient $\left(v_L, a_L\right)$ for Love-type surface waves, as provided by a method developed to yield the simultaneous solution of the period equations for wave speed (10.3.3) and attenuation coefficient (10.3.8) for any viscoelastic model ((10.4.1), (10.4.2); 2009),

(18) Propagation and attenuation vectors for component solutions in anelastic media that are inclined with respect to the surface and the surface normal, which in turn implies superimposed sinusoidal dependencies of amplitude and phase on depth that do not occur for elastic media ((10.1.13), (10.1.14), (10.2.6), (10.2.7); 2009),

(19) General sets of numerical solution curves for wave speed, attenuation coefficient, dispersion, and amplitude for Love-type surface waves versus depth and intrinsic material absorption for selected modes and intrinsic material wave speeds and periods valid for general

viscoelastic media (Figures (10.4.6), (10.4.7), (10.4.8), (10.4.9), (10.4.15); 2009),

(VII) General viscoelastic ray theory developed from first principles for horizontal and spherical media with layers and gradients as applied to forward ray-tracing problems in Chapter 11 and Appendices 12.7 and 12.8 (initially submitted to reviewers and CUP November 8, 2017 and referenced herein as this book, 2020) that revealed:

(20) Solutions of the forward ray-tracing problems for general waves in horizontal and spherical viscoelastic media with layers and gradients (Theorems (11.1.50), (11.1.53), (11.2.15), (11.3.36), (11.4.20); 2020),

(21) Phase and attenuation ray parameters $\left(p_P, p_A\right)$ unique to each general viscoelastic ray that specify the horizontal or tangential components of phase propagation and maximum attenuation at each point along the ray path as implied by generalized Snell's law for anelastic viscoelastic media (Theorems (11.1.50), (11.2.15), (11.3.36), (11.4.20); 2020),

(22) Explicit expressions for wave and ray characteristics such as wave velocity, energy flux, angle of incidence, ray path location, travel distances, Q, and amplitude attenuation that vary with changes in inhomogeneity of the wave induced by contrasts and gradients in intrinsic material absorption encountered along ray paths in anelastic media that do not occur for ray paths in elastic media (Theorems (11.1.70), (11.1.115), (11.2.30), (11.2.62), (11.3.51), (11.3.93), (11.4.29), (11.4.76), (11.4.81); 2020),

(23) Angles of incidence for critical points in phase propagation, if they exist, are in general not equal to those for reversal points in energy flux at horizontal and spherical boundaries with a contrast in anelastic intrinsic absorption, but are equal for elastic media (Theorems (11.1.166), (11.1.179), (11.3.112), ; 2020),

(24) Directions of phase propagation and maximum time-averaged energy flux are approximately equal in low-loss viscoelastic media along the entire ray path of general SII waves in horizontal and spherical media (Corollary (11.1.177), (11.2.71), (11.3.114), (11.4.85); 2020),

(25) Computation steps for forward ray tracing computer algorithms based on general viscoelastic ray theory for future application to a wide variety of problems involving anelastic media in seismology, exploration geophysics, acoustics, and solid mechanics (Tables (11.1.199), (11.5.1), (11.5.2); 2020),

(VIII) SOLUTION OF THE VISCOELASTIC REFLECTION-REFRACTION PROBLEM FOR A HEAD WAVE GENERATED BY A GENERAL SII WAVE INCIDENT AT A HEAD-WAVE ANGLE OF INCIDENCE ON A BOUNDARY IN A STACK OF HILV LAYERS (THEOREM (11.1.123); 2020) THAT REVEALED:

(26) A general viscoelastic head wave is the wave reflected at a head-wave angle of incidence for which the incident wave's horizontal phase speed equals the intrinsic wave speed of the underlying layer (Theorem (11.1.123), Figure (11.1.124); 2020),

(27) The inhomogeneous refracted wave generated at the head-wave angle of incidence at an anelastic boundary with a contrast in intrinsic absorption carries a component of energy parallel to the boundary at the intrinsic wave speed of the anelastic material, but propagates away from the boundary at a slower speed, while the waves refracted at elastic boundaries carry no energy and propagate parallel to the boundary at the intrinsic material wave speed. (Theorems (5.4.20), (5.4.21), Corollary (11.1.125), Figure (11.1.124); 1977, 2020),

(28) Expressions for travel times and amplitudes of general viscoelastic head waves that account for changes in wave speed and attenuation induced by contrasts in intrinsic material absorption consistent with those observed in many seismic refraction experiments (Figures (11.1.201), (11.1.205); 2020),

(29) Possible seismic arrivals at the surface of a stack of anelastic HILV layers due to energy transmitted across an anelastic boundary at *"wide" angles of incidence* and reflected back to the surface from deeper layers, which are arrivals that are not predicted by elastic solutions (Figures (11.1.203), (11.1.205), (11.1.206); 2020),

(IX) SOLUTION OF INVERSE PROBLEMS FOR VISCOELASTIC MEDIA (SECTIONS 11.6.1, 11.62; 2020) THAT REVEALED:

(30) Viscoelastic intrinsic material parameters $(v_{HS_m}, Q^{-1}_{HS_m})$ for multilayered horizontal media inferred from empirical travel-time and amplitude-attenuation curves and predetermined reflection-transmission coefficients (Sections 11.6.1 and 11.6.2; 2020),

(31) Viscoelastic solutions developed for the Herglotz-Wiechert integrals for horizontal and spherical media with vertical and radial material gradients (Section 11.6.2; Appendix 12.8; 2020),

(32) Viscoelastic intrinsic material parameters (v_{HS}, Q^{-1}_{HS}) for horizontal and spherical media with appropriate material gradients as a function of depth as inferred from the viscoelastic Herglotz-Wiechert integral solutions and empirical travel-time and amplitude-attenuation curves (Section 11.6.2; 2020).

These discoveries provide new insights to understand and interpret seismic waves in anelastic media such as the Earth. They explain observed characteristics of body and surface waves not explained by elastic models yet yield in the limiting case of no intrinsic material absorption the results for elastic media. They are finding application in a variety of fields, including seismology, exploration geophysics, solid mechanics, acoustics, physics, and applied mathematics as evident in the additional reading list.

The discoveries show that elastic models modified with one-dimensional attenuation approximations are adequate for some seismic problems, but may not be for other problems such as: 1) wide angle reflections and refractions beyond head-wave angles of incidence, 2) travel time and amplitude attenuation along ray paths in media with significant contrasts in intrinsic material absorption at boundaries or in gradients, 3) plane anelastic head waves, 4) wave speed and attenuation of body waves near turning points, or 5) a variety of reflection-refraction, surface wave, and multi-layer response problems in materials with a significant range in amounts of intrinsic material absorption, such as soft soils.

Continuation of this fascinating viscoelastic journey with further developments in general ray theory, computer models, and their application to account for the effects of intrinsic material absorption on waves and rays as observed with ever improving data sets is expected to reveal an exciting array of additional discoveries across a broad range of fields by a promising younger generation of scientists, mathematicians, and engineers.

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