1
One-Dimensional Viscoelasticity

The behavior of many materials under an applied load may be approximated by specifying a relationship between the applied load or stress and the resultant deformation or strain. In the case of elastic materials this relationship, identified as Hooke's Law, states that the strain is proportional to the applied stress, with the resultant strain occurring instantaneously. In the case of viscous materials, the relationship states that the stress is proportional to the strain rate, with the resultant displacement dependent on the entire history of loading. Boltzmann (1874) proposed a general relationship between stress and strain that could be used to characterize elastic as well as viscous material behavior. He proposed a general constitutive law that could be used to describe an infinite number of elastic and linear anelastic material behaviors derivable from various configurations of elastic and viscous elements. His formulation, as later rigorously formulated in terms of an integral equation between stress and strain, characterizes all linear material behavior. The formulation, termed linear viscoelasticity, is used herein as a general framework for the derivation of solutions for various wave-propagation problems valid for elastic as well as for an infinite number of linear anelastic media.

Consideration of material behaviour in one dimension in this chapter, as might occur when a tensile force is applied at one end of a rod, will introduce some of the well-known concepts associated with linear viscoelastic behaviour. It will provide a general stress-strain relation from which stored and dissipated energies associated with harmonic behaviour can be inferred for the response of an infinite number of viscoelastic models. It will permit the derivation of solutions for one-dimensional viscoelastic waves as a basis for comparison with those for two- and three-dimensional waves to be derived in subsequent chapters as initially derived by Borcherdt (1971).

1.1 Constitutive Law

A general linear viscoelastic response in one spatial dimension is defined mathematically as one for which a function $r(t)$ of time exists such that the constitutive equation relating strain to stress is given by

$$p(t) = \int_{-\infty}^{t} r(t-\tau)\, de(\tau), \tag{1.1.1}$$

where $p(t)$ denotes stress or force per unit area as a function of time, $e(t)$ denotes strain or displacement per unit displacement as a function of time, and $r(t)$, termed a relaxation function, is causal and does not depend on the spatial coordinate.

The physical principle of causality imposed on the relaxation function $r(t)$ implies the function is zero for negative time, hence the constitutive relation may be written using a Riemann-Stieltjes integral as

$$p(t) = \int_{-\infty}^{\infty} r(t-\tau)\, de(\tau) \tag{1.1.2}$$

or more compactly in terms of a convolution operator as

$$p = r * de. \tag{1.1.3}$$

Properties of the convolution operator are summarized in Appendix 1.

A corresponding constitutive equation relating stress to strain is one defined mathematically for which a causal spatially independent function $c(t)$, termed a creep function, exists such that the corresponding strain time history may be inferred from the following convolution integral

$$e(t) = \int_{-\infty}^{\infty} c(t-\tau)\, dp(\tau), \tag{1.1.4}$$

which may be written compactly in terms of the convolution operator as

$$e = c * dp. \tag{1.1.5}$$

Linear material behaviour is behaviour in which a linear superposition of stresses leads to a corresponding linear superposition of strains and vice versa. Such a material response is often referred to as one which obeys Boltzmann's superposition principle. Boltzmann's formulation of the constitutive relation between stress and strain as expressed by the convolution integrals (1.1.2) and (1.1.4) is general in the sense that all linear behaviour may be characterized by such a relation. Conversely, if the material response is characterized by one of the convolution integrals then Boltzmann's superposition principle is valid. To show this result explicitly, consider the following arbitrary linear superposition of strains

$$e(t) = \sum_{i=1}^{n} b_i\, e_i(t), \tag{1.1.6}$$

where b_i corresponds to an arbitrary but fixed constant independent of time. Substitution of this expression into (1.1.3) and using the distributive property of the convolution operator, which immediately follows from the corresponding property for the Riemann-Stieltjes integral (*see* Appendix 1), readily implies the desired result that the resultant stress is a linear superposition of the stresses corresponding to the given linear superposition of strains, namely

$$p = \sum_{i=1}^{n} b_i \, p_i = \sum_{i=1}^{n} b_i \left(r * de_i \right) = r * d\left(\sum_{i=1}^{n} b_i \, e_i \right). \tag{1.1.7}$$

Similarly, (1.1.4) implies that a linear superposition of stresses leads to a linear superposition of strains.

The term relaxation function used for the function $r(t)$ derives from physical observations of the stress response of a linear system to a constant applied strain. To show that this physical definition of a relaxation function is consistent with that defined mathematically, consider the stress response to a unit strain applied at some time, say $t = 0$, to a material characterized by (1.1.3). Specifically, replace $e(t)$ in (1.1.3) with the Heaviside function

$$h(t) \equiv \left\{ \begin{matrix} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{matrix} \right\}. \tag{1.1.8}$$

The fifth property of the Riemann-Stieltjes convolution operator stated in Appendix 1 implies that (1.1.3) simplifies to

$$p(t) = e(0+) r(t) + \int_{0+}^{t} r(t-\tau) \frac{\partial h(\tau)}{\partial \tau} d\tau, \tag{1.1.9}$$

hence,

$$p(t) = r(t). \tag{1.1.10}$$

Similarly, the creep function $c(t)$ defined mathematically may be shown to represent the strain response of a linear system to a unit stress applied at $t = 0$.

To consider harmonic behaviour of a linear viscoelastic material, assume sufficient time has elapsed for the effect of initial conditions to be negligible. Using the complex representation for harmonic functions let

$$p(t) = Pe^{i\omega t} \tag{1.1.11}$$

and

$$e(t) = Ee^{i\omega t}, \tag{1.1.12}$$

where P and E are complex constants independent of time with the physical stress and strain functions determined by the real parts of the corresponding complex numbers. Substitution of (1.1.11) and (1.1.12) into (1.1.2) and (1.1.4), respectively, shows that the corresponding constitutive relations may be written as

$$P = i\omega R(\omega) E \tag{1.1.13}$$

and

$$E = i\omega C(\omega) P, \tag{1.1.14}$$

where $R(\omega)$ and $C(\omega)$ are given by the Fourier transforms

$$R(\omega) = \int_{-\infty}^{\infty} r(\tau) e^{-i\omega\tau} \, d\tau \tag{1.1.15}$$

and

$$C(\omega) = \int_{-\infty}^{\infty} c(\tau) e^{-i\omega\tau} d\tau . \tag{1.1.16}$$

In analogy with the definitions given for elastic media the complex Modulus M is defined as

$$M(\omega) \equiv \frac{P}{E} = i\omega R(\omega). \tag{1.1.17}$$

The complex compliance is defined as

$$J(\omega) \equiv \frac{E}{P} = i\omega C(\omega), \tag{1.1.18}$$

from which it follows that the complex modulus is the reciprocal of the complex compliance, that is,

$$M(\omega) = \frac{1}{J(\omega)} \tag{1.1.19}$$

and the product of the Fourier transforms of the relaxation function and the creep function is given by the negative reciprocal of the circular frequency squared, that is,

$$R(\omega) C(\omega) = (i\omega)^{-2}. \tag{1.1.20}$$

A parameter useful for quantifying the anelasticity of a viscoelastic material is the phase angle δ by which the strain lags the stress. This phase angle is given from (1.1.17) by

$$\tan\delta = \frac{M_I}{M_R} \tag{1.1.21}$$

where the subscripts "I" and "R" denote imaginary and real parts of the complex modulus.

1.2 Stored and Dissipated Energy

Energy in a linear viscoelastic system under a cycle of forced harmonic oscillation is partially dissipated and partially alternately twice stored and returned. To account for the energy in a linear viscoelastic system under a harmonic stress excitation as characterized by a general constitutive relation of the form (1.1.13), consider the complex strain given by

$$e = Jp\,. \tag{1.2.1}$$

The time rate of change of energy in the system is given by the product of the physical stress and the physical strain rate, namely,

$$p_R\,\dot{e}_R\,, \tag{1.2.2}$$

where the dot on $\dot{e}_R$ denotes the derivative with respect to time and the subscript R denotes the real part of the strain rate. Solving (1.2.1) for p, then taking real parts of the resulting equation implies that the physical stress can be expressed as

$$p_R = \frac{J_R\,e_R + J_I\,e_I}{|J|^2}\,. \tag{1.2.3}$$

For harmonic excitation

$$e_I = -\frac{\dot{e}_R}{\omega}\,. \tag{1.2.4}$$

Substitution of (1.2.3) and (1.2.4) into (1.2.2) shows that the desired expression for the time rate of change of energy in the one-dimensional system is given by

$$p_R\,\dot{e}_R = \frac{\partial}{\partial t}\left(\frac{1}{2}\frac{J_R}{|J|^2}e_R^2\right) - \left(\frac{1}{\omega}\frac{J_I}{|J|^2}\dot{e}_R^2\right). \tag{1.2.5}$$

Integrating (1.2.5) over one cycle shows that the total rate of change of energy over one cycle equals the integral of the second term on the right-hand side of the equation. Hence, the second term of (1.2.5) represents the rate at which energy is dissipated and the first term represents the time rate of change of the potential energy in the system, that is, the rate at which energy is alternately stored and returned. The second law of thermodynamics requires that the total amount of energy dissipated increase with time, hence the second term in (1.2.5) implies that

$$J_I \le 0 \,. \tag{1.2.6}$$

A dimensionless parameter, which is useful for describing the amount of energy dissipated, is the fractional energy loss per cycle of forced oscillation or the ratio of the energy dissipated to the peak energy stored during the cycle. Integrating (1.2.5) over one cycle shows that the energy dissipated per cycle as denoted by $\Delta\mathcal{E}/cycle$ is given by

$$\frac{\Delta\mathcal{E}}{cycle} = -\pi \left|P\right|^2 J_I \,. \tag{1.2.7}$$

The first term on the right-hand side of (1.2.5) shows that the peak energy stored during a cycle or the maximum potential energy during a cycle as denoted by $\max\left[\mathscr{P}\right]$ is given by

$$\max\left[\mathscr{P}\right] = \frac{1}{2}\left|P\right|^2 J_R \,, \tag{1.2.8}$$

where $J_R \ge 0$. Hence, the fractional energy loss for a general linear system may be expressed in terms of the ratio of the imaginary and real parts of the complex compliance or the complex modulus. as

$$\frac{\Delta\mathcal{E}}{cycle}\Big/\max\left[\mathscr{P}\right] = 2\pi\frac{-J_I}{J_R} = 2\pi\frac{M_I}{M_R} \,. \tag{1.2.9}$$

Normalization of the fractional energy loss by 2π yields another parameter often used to characterize anelastic behavior, namely the reciprocal of the quality factor, which may be formally defined as

$$Q^{-1} \equiv \frac{1}{2\pi}\frac{\Delta\mathcal{E}/cycle}{\max\left[\mathscr{P}\right]} \,. \tag{1.2.10}$$

Q^{-1} for a one-dimensional linear system under forced oscillation is from (1.2.9) given by

$$Q^{-1} = \frac{-J_I}{J_R} = \frac{M_I}{M_R} \,. \tag{1.2.11}$$

Examination of (1.1.21) shows that Q^{-1} also represents the tangent of the angle by which the strain lags the stress, that is,

$$Q^{-1} = \tan\delta \,. \tag{1.2.12}$$

Another parameter often used to characterize anelastic response is damping ratio ζ, which may be specified in terms of Q^{-1} as

$$\zeta = \frac{Q^{-1}}{2}. \tag{1.2.13}$$

For the special case of an elastic system $J_I = M_I = 0$, hence $Q^{-1} = \tan\delta = \zeta = 0$.

1.3 Physical Models

The characterization of one-dimensional linear material behavior as defined mathematically and presented in the previous sections is general. The considerations apply to any linear behavior for which a relaxation function exists such that the material behavior may be characterized by a convolution integral of the form (1.1.1). Alternatively, the considerations apply to any linear material behavior for which a complex modulus exists such that (1.1.17) is a valid for characterization of harmonic behavior. Specification of the complex modulus for a particular physical model of viscoelastic behavior allows each model to be treated as a special case of the general linear formulation.

The basic physical elements used to represent viscoelastic behaviour are an elastic spring and a viscous dashpot. Schematics illustrating springs and dashpots in various series and parallel configurations are shown in Figures (1.3.5)a through (1.3.5)h. In order to derive the viscoelastic response of each configuration one end is assumed anchored with a force applied as a function of time at the other end. Forces are assumed to be applied to a unit cross-sectional area with the resultant elongation represented per unit length, so that force and extension may be used interchangeably with stress and strain.

The elongation of an elastic spring element is assumed to be instantaneous and proportional to the applied load. Upon elimination of the load the spring is assumed to return to its initial state. The constitutive equation for an elastic spring as first proposed by Hooke in 1660 is specified by

$$p = \mu e, \tag{1.3.1}$$

where μ is a constant independent of time. The assumption that the response of an elastic spring is instantaneous implies that for an initial load applied at time $t = 0$ the strain at time $t = 0$ is $e(0) = p(0)/\mu$. Hence, the creep and relaxation functions for the special case of an elastic model are $h(t)/\mu$ and $\mu h(t)$. For harmonic behavior substitution of (1.1.11) and (1.1.12) into (1.3.1) implies the complex compliance and modulus as specified by (1.1.18) and (1.1.17) are given by $1/\mu$ and μ, where the imaginary parts of each are zero. Hence, Q^{-1} as specified by (1.2.11) for an elastic model is zero.

The rate at which a viscous dashpot element is assumed to elongate is assumed to be proportional to the applied force, with the resultant elongation dependent on the entire past history of loading. The constitutive equation for a viscous element is given by

$$p = \eta \dot{e}, \tag{1.3.2}$$

where $\dot{e}$ denotes the derivative of strain with respect to time or the velocity of the elongation with respect to unit length. The viscous element is assumed not to respond instantaneously, hence its elongation due to an instantaneous load applied at time $t = 0$ is $e(0) = 0$. Integration of (1.3.2) implies the creep and relaxation functions for the special case of a viscous element are given by

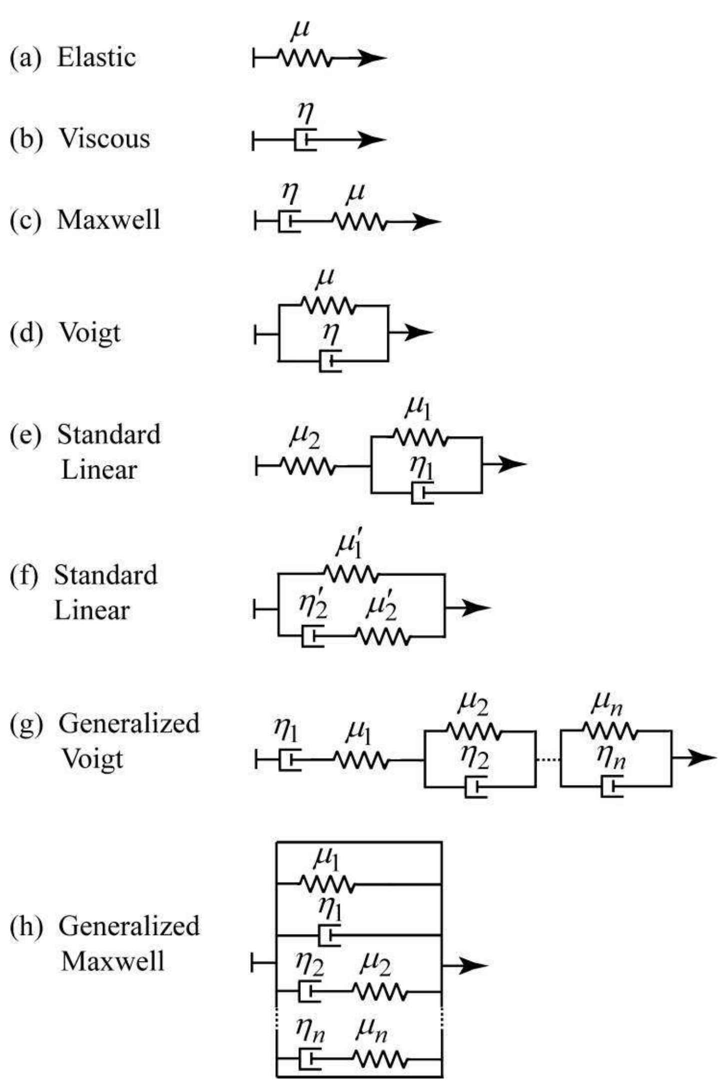

Figure (1.3.5). Schematics showing elastic spring and viscous dashpot elements in series and parallel configurations for various models of linear viscoelasticity.

$$c(t) = t\,h(t)/\eta \tag{1.3.3}$$

and

$$r(t) = \eta\,\delta(t), \tag{1.3.4}$$

where $\delta(t)$ denotes the Dirac-delta function, whose integral is unity and whose non-zero values are zero. Substitution of (1.1.11) and (1.1.12) into (1.3.2) implies the complex compliance and modulus for a viscous element are $J(\omega) = 1/(i\omega\eta)$ and $M(\omega) = i\omega\eta$. Equation (1.2.11) implies Q^{-1} is infinite, because no energy is alternately stored and returned in a viscous element.

An infinite number of viscoelastic models may be derived from various serial and parallel configurations of elastic springs and viscous dashpots. Schematics for common models are shown in Figure (1.3.5). Three fundamental viscoelastic models are the Maxwell model, which assumes the basic elements are in series, the Voigt model, which assumes the basic elements are in parallel, and a Standard Linear model, which assumes a spring in series with a Voigt element or a spring in parallel with a Maxwell element. Generalizations of these models are the Generalized Voigt model and the Generalized Maxwell model. The Generalized Voigt model includes a Maxwell model in series with a sequence of Voigt elements in series. The Generalized Maxwell model includes a Voigt element in parallel with a sequence of Maxwell elements in parallel. A Standard Linear model may be considered as a special case of a Generalized Voigt model with $n = 2$ and $\eta_1^{-1} = 0$.

The two configurations shown for a Standard Linear model (Figures (1.3.5)e and f) are equivalent in that the parameters of the elements may be adjusted to give the same response. Similarly, the configuration of springs and dashpots for any model involving more than two of these elements is not unique, in that other configurations of springs and dashpots in series and parallel will yield the same response.

For the Maxwell model the strain resulting from an applied load is the sum of the strains associated with the individual elements in series. Hence, the resultant strain rate is given by

$$\dot{e} = \dot{e}_1 + \dot{e}_2 = \frac{\dot{p}}{\mu} + \frac{p}{\eta}, \tag{1.3.6}$$

where the initial strain, as implied by the assumed instantaneous response of the spring, is $e(0) = p(0)/\mu$. Integration of (1.3.6) and substitution of a unit stress implies that the creep function for a Maxwell model is

$$c(t) = \left(\frac{1}{\mu} + \frac{1}{\eta}t\right)h(t). \tag{1.3.7}$$

Similarly, integration and substitution of a unit strain implies the relaxation function for a Maxwell model is

$$r(t) = \mu \exp\left[-(\mu/\eta)\, t\right] h(t). \tag{1.3.8}$$

Substitution of (1.1.11) and (1.1.12) into (1.3.6) together with (1.1.17) and (1.1.18) implies that the complex compliance and complex modulus for a Maxwell model are given by $J(\omega) = 1/\mu - i/(\omega\eta)$ and $M(\omega) = \left(1/\mu - i/(\omega\eta)\right)^{-1}$, from which (1.2.11) implies $Q^{-1} = \mu/(\omega\eta)$.

For a Voigt model the applied stress is the sum of the stress associated with each of the elements in parallel. Hence the applied stress is given by

$$p = \mu e + \eta \dot{e}, \tag{1.3.9}$$

where the initial strain is $e(0) = 0$. The creep and relaxation functions inferred from (1.3.9) for the Voigt model are

$$c(t) = \frac{1}{\mu}\left(1 - \exp\left[-(\mu/\eta)\, t\right]\right) h(t) \tag{1.3.10}$$

and

$$r(t) = \eta\,\delta(t) + \mu\, h(t). \tag{1.3.11}$$

Substitution of (1.1.11) and (1.1.12) into (1.3.9) implies the complex compliance and modulus for a Voigt model are $J(\omega) = 1/(\mu + i\omega\eta)$ and $M(\omega) = \mu + i\omega\eta$. Hence, (1.2.11) implies $Q^{-1} = \omega\eta/\mu$.

For a Standard Linear model with an applied load, the resultant strain is the sum of the strains associated with the spring in series with the Voigt element, while the applied stress is the same for the spring and Voigt elements in series. The resulting equations for configuration "e" shown in Figure (1.3.3) are

$$p = \mu_1 e_1 + \eta_1 \dot{e}_1 = \mu_2 e_2 \tag{1.3.12}$$

and

$$e = e_1 + e_2, \tag{1.3.13}$$

which upon simplification may be written as

$$p + \tau_p \dot{p} = M_r \left(e + \tau_e \dot{e}\right), \tag{1.3.14}$$

where τ_p is the stress relaxation time under constant strain defined by

$$\tau_p \equiv \frac{\eta_1}{\mu_1 + \mu_2}, \tag{1.3.15}$$