PART I

GETTING STARTED
Before we begin discussing the theory we must agree on some notation. The space–time coordinates are represented by the Lorentz four-vector $x^\mu = (x^0, x^i) \equiv (ct, \vec{x})$ where $i = 1, 2, 3$. The index $\mu$ takes on values $0, 1, 2, 3$. The index $\mu$ can be raised or lowered using the space–time metric $g_{\mu\nu}$ with $x_\mu \equiv g_{\mu\nu} x^\nu$ and

$$g_{\mu\nu} \equiv \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \text{ in Cartesian coordinates.} \quad (1.1)$$

We define the Lorentz-invariant four-vector length by

$$x^2 \equiv x^\mu x^\nu g_{\mu\nu}. \quad (1.2)$$

The energy–momentum four-vector is given by $P^\mu = (P^0, \vec{p} c) \equiv (E, \vec{p} c)$ and the particle mass is given via the Lorentz scalar

$$P^2 \equiv P^\mu P_\mu \equiv E^2 - \vec{p}^2 c^2 = m^2 c^4. \quad (1.3)$$

In order to discuss dimensions of various quantities, it is convenient to consider the Dirac equation, which describes the relativistic quantum mechanics of spin $\frac{1}{2}$ particles. We first introduce the Dirac gamma matrices, satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$ 

Then Dirac defined the Hamiltonian given by the $4 \times 4$ matrix

$$H = \vec{\alpha} \cdot \vec{p} c + \beta mc^2 + V(\hat{x}), \quad (1.4)$$

where $\gamma^0 \equiv \beta \gamma^i = \beta \alpha^i$.

The energy eigenvalue is given by

$$E^2 = H_0^2 \equiv \vec{p}^2 c^2 + m^2 c^4, \quad (1.5)$$

where $H_0$ is the free particle Hamiltonian. In the Schrödinger equation, the energy-momenta are replaced by operators in Hilbert space. We have

$$P^\mu = (P^0, \vec{p} c) \equiv i\hbar \frac{\partial}{\partial x^\mu} = i\hbar \frac{\partial}{\partial x^\mu},$$

$$(\text{with } \nabla^i \equiv \frac{\partial}{\partial x^i}). \text{ The potential for an electron in the field of } Z \text{ protons is given by}$$

$$V = e\phi = -\frac{Ze^2}{4\pi\epsilon_0 r} \text{ central force.} \quad (1.6)$$
Finally, we obtain the Schrödinger equation
\[ H \psi = i \hbar \frac{\partial}{\partial t} \psi, \]  
(1.7)
\[ \hbar c \left[ \alpha \cdot (-i \nabla) + \beta \frac{mc}{\hbar} - \frac{Z \alpha}{r} \right] \psi = i \hbar \frac{\partial \psi}{\partial t}, \]  
(1.8)
where we define the dimensionless fine structure constant
\[ \alpha \equiv \frac{e^2}{4 \pi \varepsilon_0 \hbar c}. \]

We can now consider the dimensions of various quantities. Each term in the above equation necessarily has the same dimension. We have, where the bracket \( [x] \) represents the dimension of the quantity \( x \) and \( \ell \) stands for length,
\[ [\nabla] = \ell^{-1}, \]
\[ [\frac{\hbar}{mc}] = \ell \quad \text{Compton wavelength} \]
\[ [x] = \ell \quad [\alpha] = \ell^0 \]
\[ [ct] = \ell \]
\[ [E] \sim \hbar c \ell^{-1} \quad [p] = \hbar \ell^{-1}. \]
\[ [m] \sim \frac{\hbar}{c} \ell^{-1}. \]

We will define the unit of energy commonly used in atomic, nuclear, astro and particle physics. We have the electron volt given by 1 eV = 1.6 × 10^{-19} joules, where 1 joule = 1 coulomb volt. Since the charge on an electron, \( e = 1.6 \times 10^{-19} \) coulombs, 1 eV is, by definition, the energy an electron has after traversing a voltage difference of 1 volt. Then 1 keV = 10^3 eV; 1 MeV = 10^6 eV; 1 GeV = 10^9 eV: and 1 TeV = 10^{12} GeV.

Note, also in the mks system we have
\[ 1 \text{ kg} \frac{m^2}{s^2} \equiv 1 \text{ joule} \]
\[ \Rightarrow 1 \text{ kg} = (3 \times 10^8) \frac{\text{joule}}{c^2} \]
\[ = \frac{9 \times 10^{16} \text{ MeV}}{1.6} \frac{10^9 \text{ MeV}}{c^2} = 5.625 \times 10^{29} \frac{\text{MeV}}{c^2}. \]

In these units the proton and electron masses are given by
\[ m_p \simeq 1.67 \times 10^{-24} \text{ gm} \simeq 1.67 \times 10^{-27} \text{ kg} \]
\[ \simeq 938 \text{ MeV}/c^2 \]
\[ m_e \simeq 9 \times 10^{-31} \text{ kg} \simeq 50 \times 10^4 \text{ eV}/c^2 \]
\[ \simeq \frac{1}{2} \text{ MeV}/c^2. \]

Note, given \( e = 1.6 \times 10^{-19} \) C and \( 1 \frac{\text{Nm}}{\text{C}^2} = 9 \times 10^9 \) N m^2/C^2, we find an approximate value for the fine structure constant, \( \alpha \approx \frac{e^2}{4 \pi \varepsilon_0 h c}. \) Given fixed values of \( e, \hbar \) and \( c, \) better experimental measurements of \( \alpha \) can determine \( \varepsilon_0 \) to higher accuracy.
The values of $\hbar$ and $c$ will be important to remember. We have $h = |E \cdot t| = 6.6 \times 10^{-22} \text{ MeV} \cdot \text{s}$ and $c = 3 \times 10^8 \text{ m/s}$. Thus we have $h \cdot c \approx 200 \times 10^{-13} \text{ MeV cm}$. Define the unit of length 1 fermi $\equiv 1 \text{ fm} = 10^{-13} \text{ cm}$ and we have $h \cdot c \approx 200 \text{ MeV fm}$.

Cross-sections, $\sigma$, are in units of a barn $\equiv 1 \text{ b} = 10^{-24} \text{ cm}^2$. Strong-interaction cross-sections $\sigma_{\text{str}} \sim 30 \text{ mb} = 3 \times 10^{-26} \text{ cm}^2$ and weak cross-sections are of order $\sigma_{\text{weak}} \sim 1 \text{ picobarn} \equiv 1 \text{ pb} = 10^{-36} \text{ cm}^2$. Finally a femtobarn $\equiv 1 \text{ fb} = 10^{-39} \text{ cm}^2$.

Note, following the standard conventions, in this book we will redefine the dimensions of space and time such that $h = c = \epsilon_0 = 1$. We can always revert to the mks units by rescaling the end result by the appropriate powers of $h$ and $c$. Therefore $(200 \text{ MeV})^{-1} \equiv 1 \text{ fm} \equiv 3 \times 10^{-24} \text{ s}$, where the notation $\equiv$ means conversions using $h$ and $c$. Finally, a cross-section of $(200 \text{ MeV})^{-2} \equiv 10 \text{ mb}$ and $1 \text{ TeV}^{-2} \simeq 400 \text{ pb}$.