Introduction

Abstract mathematics, from its earliest times in ancient Greece right up to the present, has always presented a major challenge for philosophical understanding. On the one hand, mathematics is widely considered a paradigm of providing genuine knowledge, achieving a degree of certainty and security as great as or greater than knowledge in any other domain. A part of this, no doubt, is that it proceeds by means of deductive proofs, thereby inheriting the security of necessary truth preservation of deductive logical inference. But proofs have to start somewhere: ultimately there need to be axioms, and these are the starting points, not end points, of logical inference. But what then grounds or justifies axioms? The question becomes especially urgent when it is considered that the subject matter of pure mathematics, including its axioms, apparently consists of *abstracta* such as numbers, functions, classes, and relations, which are non-spatiotemporal and do not enter into causal interactions (with us or anything else). This is true even for Euclidean geometry, which originally was conceived as investigating properties of actual physical space and time, but nevertheless treats directly of dimensionless points, perfectly straight lines of 0 breadth, ideal perfect figures such as triangles and rectangles, etc., none of which exist in the material world.

In our own recent history, the logical empiricists, led by Rudolf Carnap and inspired by Gottlob Frege, proffered the doctrine of *analyticity*, that mathematical axioms are "analytic," but now in the sense of "guaranteed true solely in virtue of meanings of terms," or "true entirely by linguistic convention."[1] This was regarded as compatible with the obvious fact that theorems could be highly surprising and informative, as logical deductions can be highly complex and intricate, so that logical consequences of given axioms may appear quite unpredictable.

[1] Though Frege explicitly declared in the *Grundlagen der Arithmetik* that (in effect, second-order) arithmetic was "analytic," this was explained as "derivable from logic," whereas Frege's "logic" was nowhere said to be "true by linguistic convention." Indeed, not only was that logic committed to many infinitistic functions, it was also proved inconsistent due to implying Russell's paradox.

Now, as is well known, the doctrine of analyticity was severely critiqued by W. V. Quine (and others), due to the questionable scientific status of the concept of linguistic "meaning." But even granting – for the sake of argument – the scientific status of meaning concepts, there remains the problem that at least some of the axioms of going mathematics just seem not to qualify as "analytic" at all. Famously, the Euclidean parallels postulate (EPP) seems not to qualify, resisting attempts to derive it from anything more basic. And the case seemed to be sealed when, in the nineteenth century, it was discovered that the EPP could consistently be negated, giving rise to genuine non-Euclidean geometries, provably consistent relative to Euclidean geometry.

Now it may occur to the reader that one could reinterpret the EPP as claiming only what is true in a genuinely Euclidean space, thereby ensuring its analyticity, as a space would be "Euclidean" only if it satisfied the EPP. But now we have to accept the existence of a (possibly non-physical) Euclidean space, and what yields the analyticity of *that*? Indeed, that involves the existence of an infinite totality, and how can that be guaranteed true solely by the meanings of words?

Indeed, for another striking example of a non-analytic axiom, consider the Axiom of Infinity, that there exists an infinite set. Any attempt to derive this appears quite circular. For example, Dedekind thought he could derive it from reflecting on his capacity to entertain the thought of any thought that he could entertain. But this failed, not only for the reason that it is dubious that we can even understand enough iterates of "the thought of . . . the thought of my own ego (or whatever the initial object may be)," but because Dedekind needed to assume that "all objects that could be objects of [his] thought" form a "system" or set. But what guarantees this? Perhaps all subsets of objects of his thought could be objects of his thought, in which case Cantor's theorem (that there are always more subsets of a given set than members of it) would rule out that all objects of his thought form a set, as that set would have all its subsets as members, which is a contradiction!

The famous Axiom of Choice of set theory provides another example of an axiom whose "analyticity" seems impossible to secure.[2]

Such examples led to a view known as "if-thenism" or "deductivism" (espoused by Russell [1903], and later, at least in part, by Hempel [1945]), according to which mathematics need not assert its axioms and can confine itself to conditional claims of the form, "If these axioms, then this theorem." (A recent version of this is examined critically in some detail in the essay of Chapter 14.)

[2] The Axiom of Choice says that, given any set S of non-empty sets, s, there exists a "choice function" f on S, i.e. such that the value $f(s)$ of f is a member of s. Such a choice function on any infinite S "does an infinite amount of choosing at once" and is thus itself an infinitistic object.

Thus, even in the late twentieth century, the challenge regarding philosophical understanding presented by mathematics was far from being met. One overarching question addressed by a number of the essays of this collection is to what extent the rise of modern structuralism – in particular modal-structuralism (MS), begun by Putnam [1967] and further developed by me (Hellman [1989] and Hellman [1996], reproduced as Chapter 1 in this volume) – makes significant progress toward meeting this challenge.

The crucial starting point of modal-structuralism is to adopt the Dedekindian-Hilbertian view of mathematical axioms as *defining conditions* on types of mathematical structures of interest, rather than as asserted truths outright (as in the traditional Euclidean-Fregean view). The next step is to interpret ordinary mathematical statements *S* of a branch of mathematics as stating what would necessarily hold of structures of the appropriate type that there might be (logically speaking), structures characterized by the axioms of the branch of mathematics in question. Thus *S* is construed as a quantified modal conditional whose antecedent is the conjunction of the relevant axioms (relativized to an arbitrary given domain) and whose consequent is *S* (with its quantifiers restricted to the given domain). If that were all, then we would have a second-order logical version of if-thenism; and then there would be the problem that, if the axioms were inconsistent, then any such conditional would count as true or valid, regardless of the consequent – an intolerable situation. In order to block this "problem of vacuity," a further step is required, viz. to assert the (second-order logical) possibility of there being a structure fulfilling the relevant axioms (hence ruling out inconsistency), what we have called "the categorical component" of a modal-structural interpretation (the quantified conditionals constituting "the hypothetical component"). And it is just here that the logicist idea, that all mathematical truths are analytic, breaks down, for these modal existence postulates cannot be determined true solely in virtue of the meanings of the words involved. (Compare the ontological argument, which purports to "prove" the existence of a necessary, perfect being by counting existence as a perfection! True enough, *if* G is a necessary, perfect being, *then* G exists. But nothing guarantees that there actually *are* any necessary, perfect beings. Moral: you cannot *define* objects into existence!)

At the time that I was developing MS, I was unaware of the full potential of some crucial logical machinery already developed by Boolos [1985], that of plural quantification. Specifically, as shown by Burgess, Hazen, and Lewis [1991] (BHL) in their Appendix to Lewis [1991], the *combination* of plural quantification and atomic mereology (but neither separately) enables explicit general constructions of ordered-pairing, which achieves the expressive power of a full classical theory of relations (polyadic second-order logic). In the absence of this, Hellman [1989] got by with some rather ad hoc machinery to achieve the requisite expressive power to carry out MS interpretations. But the

BHL constructions afford a more general and smoother development, as explained in the first essay of this volume, "Structuralism without Structures," Chapter 1.

As the reader may well know, there are several other structural approaches to mathematics besides modal-structuralism. Indeed, set theory and category theory have each provided such from within mathematics itself. (For a systematic comparison of these along with Shapiro's *sui generis* structuralism, see Hellman and Shapiro [2019], also Hellman [2005].) Set-theoretic structuralism (STS), based on model theory, is probably the best known of all, and for much mathematics it does very well. There are, however, two main problems with it. First, it fails to treat set theory itself structurally, despite the fact that there are multiple, conflicting but perfectly legitimate set theories deserving of recognition. But second, and most important here, STS based on the Zermelo–Fraenkel axioms, with Choice (ZFC), as usually understood, is committed to a unique, maximal universe of "all sets," despite the fact that the very notion of "set" is indefinitely extensible, as are the notions of "ordinal" and "cardinal." That is, in the case of "set," by our understanding of sets as subject to certain operations, any totality of sets can be transcended by means of those operations, for instance the operation of forming singletons or forming powers (that is, forming the totality of all subtotalities). Thus, it is a central postulate of the modal-structural interpretation of set theory that any domain of sets (or, more neutrally, "set-like objects," objects obeying the axioms of ZFC) can be properly extended to a more comprehensive domain. We call this the Extendability Principle (EP). This applies also to category theory's (CT's) version of set theory, as explained in Chapter 2, and it rules defective commitment to a category of "all sets."

The essay of Chapter 2, "What is categorical structuralism?" assesses responses by Colin McLarty and Steve Awodey to my earlier critique of category theory's approach to mathematical structuralism.[3] There I had complained about the lack of assertory axioms governing existence of category-theoretic structures, and McLarty had countered that two axiom systems due to Lawvere met my concerns, axioms on the (sic) category of sets and axioms on the (sic) category of categories. On the other hand, Awodey had maintained that such axioms are unnecessary, that CT can get by with an entirely schematic approach to mathematical structures. The essay of Chapter 2 exposes a common problem with both of these approaches, viz. that both implicitly rely on some concept of *satisfaction* (of sentences by systems of objects), usually articulated via set theory, although second-order logic can be used

[3] References to McLarty and Awodey are given in Chapter 2.

instead. Thus CT's autonomy, as a foundational framework, from set theory or second-order logic has yet to be established by its proponents.[4]

Returning now to our theme of indefinite extensibility: in contrast to both set theory and category theory, as usually understood, the MS approach adopts a "height-potentialist" perspective, based on the EP framed modally. As Putnam [1967] forcefully put it, "Even God could not make a model of Zermelo set theory that it would be *mathematically* impossible to extend" (p. 310). But since MS allows modal quantification over arbitrary set-like objects – for example "for any set that there might be, there would also be its power set," etc. – what blocks commitment to a totality of "all possible sets or set-like objects"? Since MS eschews possible worlds or *possibilia* of any sort, the answer is that collections can only contain what would exist under given circumstances, not anything that merely *might* then have existed. Invoking "worlds" as heuristic only, we can say that sets or collections or set-like objects are "world-bound." It is impossible to form collections "across worlds." It literally makes no sense to speak of "the collection of all possible set-like objects."[5] Thus, in contrast to Zermelo's [1930] effort to articulate a height-potentialist view (which did not employ modal operators), MS naturally avoids commitment to "proper classes" or "ultimate infinities" in an absolute sense. The notion of "proper class" can only be *relative to a domain*: what qualifies as a proper class (hence not a member of anything) relative to domain D, functions as a *bona fide* set relative to any possible proper extension of D; and there can be no "union of all possible domains."[6]

The essays of Chapters 3 and 4 develop some important consequences of the height-potentialist view just sketched. Chapter 3 describes the MS resolutions of the set-theoretic paradoxes, concentrating on the so-called Burali-Forti paradox, of "the largest ordinal." There it is shown how, in a potentialist sense, MS can respect the desideratum that any well-order relation whatever can have an ordinal representing it. This is in contrast to standard resolutions of Burali-Forti based on a single fixed universe of "all sets." Chapter 4 then shows how a natural modal principle on the extendability of "stages" of sets on the well-known iterative conception of set leads to new derivations of the axioms of Infinity and Replacement, not available to Boolos' original [1971]

4 Our analysis thus sustains the well-known earlier analysis by Feferman [1977], but focuses on the problem of articulating structuralism rather than foundations generally.

5 Here we follow Kripke's [1980] "actualist" conception of the alethic modalities, as contrasted with Lewis' [1986] "possibilist" or "modal-realist" conception.

6 While Zermelo [1930] did clearly state that the "set/proper class" distinction is relative to a domain, still that work is naturally formalized in axiomatic second-order logic, an axiom of which guarantees a class of all members of any domains, an ultimate proper class, contrary to Zermelo's (non-modal) Extendability Principle. In avoiding this "explosion," modality does essential conceptual work for the MS interpretation.

stage theory, or any other height-actualist theory recognizing a plurality of "all stages."

Thus far, we have seen that MS requires as assertory axioms statements affirming the possibility of structures of the appropriate type, geared to the mathematical axioms defining that type of structures. Thus, both Hilbertian-style and Fregean-style axioms are needed. But we have not yet said anything explicitly about the ontological commitments of mathematical theories interpreted according to MS. Take the most elementary case of arithmetic. Does the MS interpretation quantify over numbers as objects? No; all it requires is that there possibly be a progression; it makes no difference at all what objects make up the progression. All that matters is that, whatever the objects, they be arranged in the right way, as required by the Dedekind–Peano axioms. Thus, on interpretation, the predicate "__is a natural number" is *eliminated*. Similarly for the integers, the rationals, the reals, and the complexes. MS is, after all, "mathematics without numbers" (as explained in detail in Hellman [1989]). Now one of the virtues of the BHL machinery, deploying the logic of plurals combined with atomic mereology, is that it allows us to eliminate even *structures* as objects, in favor of speaking directly of enough objects – of whatever sort – interrelated in the right ways, as dictated by the proper mathematical axioms. Thus, as indicated in the title of Chapter 1, we have a "Structuralism without Structures." This thus raises the prospect that MS may be fully nominalistic, at least in the sense that abstract entities need not be recognized. To a surprising degree, this is correct. But, as will now be explained, it is not *entirely* correct.

Suppose we begin with a postulate asserting the possibility of a countable infinity of mereological atoms, say, satisfying the Dedekind–Peano axioms for natural numbers. Such objects can readily be conceived as part of space or space-time, for example non-overlapping space-time regions. They need not be abstract. Then applying mereology, we have all wholes of such atoms, also not abstract. This gives us a continuum of concrete objects, at the level of the classical real numbers, or classical second-order number theory, arguably enough to support virtually all of scientifically applicable mathematics. But we can go even further, using plural quantification over these real-number surrogates, yielding the equivalent of full, classical third-order number theory, again within the confines of nominalism. Furthermore, now consider that we could start off with a *continuum* of atoms instead of a countable infinity of such. Then applying mereology we obtain all wholes of such atoms, corresponding to all non-empty subsets of the atoms. Finally, with plural quantification over such wholes, we attain the level of third-order real analysis or fourth-order number theory, all within a nominalist framework. Thus, vast amounts of pure and applied mathematics (including e.g. differentiable geometry of Riemannian manifolds, measure theory, and much more) are nominalistically reducible.

(Indeed, we do not even need to invoke modality for this much, as actual space-time regions furnish us with enough objects.)

Such constructions thus serve to undermine the well-known indispensability arguments for the need to recognize mathematical *abstracta* in order to do justice to scientific applications of mathematics. In this sense, a nominalist ontology is enough to support virtually all such applications, depriving platonism of arguably its most powerful argument. Such considerations led us to reassess nominalism in the essay of Chapter 5. They are there coupled with an effort to reformulate nominalism as an epistemic thesis, rather than a strictly ontological one, viz. that there is no compelling evidence or reason for invoking mathematical *abstracta*, appearances to the contrary notwithstanding.

There is, however, a consideration that suggests that that conclusion may go too far. That is that the above constructions do not extend to abstract set theory, even Zermelo set theory, not to say ZFC. While the modal-structural interpretations of those set theories do eliminate the predicate ‘is a set’, much as ‘is a number’ is eliminated, still the postulate of the possibility of enough objects to form a model of those set theories goes well beyond the reaches of nominalist ontology, as described above. Not all such objects could be conceived as part of space-time, even a space-time of higher dimension. (Even postulating a continuum of dimensions would not take one far enough!) Yet, as the work of Harvey Friedman suggests, abstract set theory may well be required to solve problems at the level of sets of integers (see the essay, “On the Gödel–Friedman program,” of Chapter 10). It is not inconceivable that such problems might even arise within physics. In that case, the Quine–Putnam indispensability argument could be restored. But for now that remains quite speculative. Thus, the thrust of “On nominalism” is that at present, except for achieving a realist understanding of higher set theory, a nominalist ontology qualifies as a default position, trading places with platonist ontologies that have dominated in the past.

Finally, on the topic of nominalism, the essay of Chapter 6, “Maoist mathematics?”, which is a critical study of Burgess and Rosen’s [1997] book, *A Subject with No Object: Strategies for Nominalist Interpretation of Mathematics*, defends nominalistic reconstruction programs against the charge of facing a dichotomy of either proposing (unjustifiably) to uncover the “real, deeper meaning” of mathematical theories, or (recklessly) advocating a revolutionary revision of mathematics as practiced. We argue that this is a false dichotomy, that the nominalist programs considered by Burgess and Rosen – those due to Field, Chihara, and Hellman – are proposed as neither “hermeneutic” nor “revolutionary,” but rather serve as *rational reconstructions* designed to mitigate epistemological and metaphysical problems confronting platonistically construed mathematics.

This brings us to Part II, which consists mainly of essays on predicative mathematics, two of which were done jointly with Solomon Feferman. Now predicative foundations grew out of the work of Russell, viz. his ramified type theory, and writings of Poincaré and Weyl, and was designed to avoid so-called impredicative definitions or specifications of sets, that is specifications by formulas with quantifiers ranging over totalities which include the very set being introduced. For example, consider the classical least upper bound principle governing sets of real numbers. This says that any non-empty bounded set of reals has a least upper bound (or greatest lower one). Specification of such a bound involves a quantifier ranging over all (upper) bounds of the given set, which of course includes the least bound. Thus the principle is called "impredicative." Now classical set theorists have no problem with such specifications, as they regard the totalities of sets involved as objectively existing, independent of their being picked out by our languages. But those with constructivist inclinations find such specifications viciously circular (recall Russell's "vicious circle principle," essentially a ban on impredicative specifications of sets or other objects). Ultimately, this traces back to skepticism over the power-set operation, passing from an infinite set to the set of *all* of its subsets. Instead, the predicativist restricts this operation to taking the set of all *definable* subsets of the given (infinite) set, where the definitions lack quantifiers ranging over the very subset being specified.

As another paradigmatic example, consider the classical Fregean and Dedekindian specifications of the totality of natural numbers as the intersection of all classes containing the initial number (0 for Frege, 1 for Dedekind) and closed under the successor operation. Again, this reference to "all classes" includes the very class being introduced; hence the specification is impredicative.

The effect of this ban on impredicative specifications of sets is to avoid uncountably infinite sets, in an absolute sense, as formulas of countable languages are required to specify predicatively subsets of an infinite set. Instead, the predicativist recognizes a kind of *relative* uncountability of the real numbers, based on the negative conclusion of Cantor's diagonalization argument: given any putative enumeration of all the reals, the diagonal argument produces a real that differs from each real of the given enumeration. So far so good. But this is just interpreted to mean that more reals need to be recognized at any stage of construction. The continuum is thus viewed as an incompleteable, indefinitely extensible totality, something like what even some (but not all) classical set theorists recognize as true of the putative universe of "all sets or ordinals."

Two more features of predicative mathematics need to be mentioned here by way of background. The first is that classical logic is accepted, distinguishing predicative from constructive mathematics with its renunciation of the law of excluded middle (framed either classically or with intuitionistic logical connectives). Predicative mathematics is thus known as "semi-constructivist." Second, as usually presented, predicative mathematics begins by taking the natural numbers as given. Predicativity is understood as *relative to the natural numbers*. Thus, Poincaré's misgivings concerning logicism (which sought a logical foundation of arithmetic) are taken to heart: the natural-numbers system is regarded as more fundamental than the full battery of logicist machinery (which included second- and higher-order logical notions, not merely first-order ones).

There arises here, however, a nagging question: given that classical logicist foundations of arithmetic had to resort to impredicative definitions to obtain a natural-numbers structure, as reviewed above, would it not be better for predicative mathematics to begin without taking that structure for granted, but somehow to *derive* it (its existence)? After all, it is an infinite structure of a special type.

As had been pointed out by Dan Isaacson [1987], the framework of "weak second-order logic," with axioms quantifying over *finite sets* as well as individuals, does permit a characterization of a natural-numbers structure, unique up to isomorphism. This suggested to me that it should be possible for predicative mathematics to begin with an elementary theory of finite sets and build up a natural-numbers structure *predicatively relative to the notion "finite set."* Extensive correspondence with Solomon Feferman, leading proof theorist well known for developing predicative mathematics, then resulted eventually in the two papers reprinted here as Chapters 7 and 8. Notably, the theory of finite sets developed there is quite weak, lacking any axiom of finite-set induction, thereby avoiding the charge of circularity of our construction, which effectively derives mathematical induction governing natural numbers. Furthermore, our derivation of existence and unicity of a natural-numbers structure brings out an important difference between the notion of "finite set" as compared with "natural number," namely that finite sets are "self-standing" rather than inherently part of a structure, whereas the opposite is true of "natural number" (especially in Dedekind's sense of "finite ordinal"). In our view, it makes no sense to consider a natural number in isolation from a structure of at least a segment of natural numbers; whereas reference to finite sets makes sense apart from their belonging to a structure of, say, hereditarily finite sets (of some given individuals) ordered by set-inclusion. The upshot is that we provide a predicativist-logicist foundation of arithmetic, thereby meeting Poincaré's challenge.

This brings us to the essay of Chapter 9, "Predicativism as a Philosophical Position," where we assess the philosophical import of predicative foundations. Various limitative theses are examined and found wanting, mainly because their very assertion requires transcending the limits of predicativist mathematics. Instead, we find the main contributions of predicative foundations to be in the area of mathematical epistemology, through its detailed examination of "what rests on what" and "why a little bit goes a long way" (both of which phrases are titles of insightful papers of Feferman).[7] Unlike radical constructivism (examined in Part III), predicativism does not purport to set limits to classical mathematics, but rather seeks to show the sufficiency of predicative methods for the vast bulk of scientifically applicable mathematics. It thus poses a major challenge to the Gödel–Friedman program, which seeks to justify abstract set theory as needed to solve ordinary (or ordinary-appearing) mathematical problems.

The main point made in the last essay (Chapter 10) of Part II, "On the Gödel–Friedman program," is that Bayesian confirmation theory is relevant to meeting this challenge posed by predicative foundations. As explained in that essay, there is a major gap between statements of consistency of large cardinals and statements asserting directly the mathematical existence of such cardinals. Yet it is the former, not the latter, that recent work of Friedman demonstrates equivalent to certain low-level combinatorial statements very similar to statements that are provable without higher set theory (as in Friedman's Boolean Relation Theory (BRT)). As it stands, predicative mathematics is adequate for proving the statements of BRT that have been shown equivalent to statements of consistency of certain large cardinals. However, from the standpoint of ordinary mathematical practice, such consistency statements are arcane renderings of metamathematical content of precisely the kind that the Gödel–Friedman program seeks to improve upon. Our essay sketches how, in principle, a kind of inductive evidence can be gained to support the assertions of mathematical existence of the relevant large cardinals. The latter can thus emerge as the *best explanation* for a variety of independently justifiable consistency statements, in line with Gödel's ideas set out in his well-known [1947] paper, "What is Cantor's continuum problem?"

Turning to the essays of Part III, these focus on various logical systems used in different approaches to mathematics and its foundations. Now, to one who shares the popular misconception of mathematics as a cut-and-dried discipline of universally agreed upon results, it may come as something of a shock to learn that there are actually vastly different "schools" of mathematics favoring even different logics. But that is actually the case, as manifested in the divergence between mainstream classical mathematics based on (first- or higher-order) classical logic on the one hand, and, on the other, various versions of constructive mathematics based on intuitionistic logic, well known for renouncing certain classical logical laws, especially the law of excluded middle (that either p or not-p holds for any mathematical sentence, p) and proof of existence by

[7] See Feferman [1998].