I

QUANTUM FIELDS, GENERAL FORMALISM, AND TREE PROCESSES

1 Review of Classical Field Theory: Lagrangians, Lorentz Group and its Representations, Noether Theorem

In this book, as I have mentioned, I will assume a knowledge of classical field theory and quantum mechanics, and I will only review a few notions from them, immediately useful, in the first two chapters. In this first chapter, I will start by describing what quantum field theory is, and after that I will review a few things about classical field theory. In the next chapter, a few relevant notions of quantum mechanics, not always taught, will be described.

Conventions I will use theorist's conventions throughout, with $\hbar = c = 1$, which means that, for example, $[E] = [1/x] = 1$. I will also use the *mostly plus* metric, for instance in $3+1$ dimensions with signature $-+++$.

1.1 What is and Why Do We Need Quantum Field Theory?

Quantum mechanics deals with the quantization of particles, and is a nonrelativistic theory: time is treated as special, and for the energy we use nonrelativistic formulas.

On the contrary, we want to apply *quantum field theory*, which is an application of quantum mechanics, to *fields* instead of particles, and this has the property of being *relativistic* as well.

Quantum field theory is often called (when derived from first principles) *second quantization*, the idea being that:

- The *first* quantization is when we have a single particle and we quantize its behavior (its motion) in terms of a wavefunction describing probabilities.
- The *second* quantization is when we quantize the wavefunction itself (instead of a function now we have an operator), the quantum object now being the number of particles the wavefunction describes, which is an arbitrary (variable) quantum number. Therefore, the field is now a description of an arbitrary number of particles (and *antiparticles*), and this number can *change* (i.e. it is not a constant).

People have tried to build a *relativistic quantum mechanics*, but it was quickly observed that if we do that, we cannot describe a single particle:

- First, the relativistic relation $E = mc^2$, together with the existence (experimentally confirmed) of *antiparticles* which annihilate with particles giving only energy (photons), means that if we have an energy $E > m_p c^2 + m_{\bar{p}} c^2$, we can create a particle–antiparticle pair, and therefore the number of particles cannot be a constant in a relativistic theory.

- Second, even if $E < m_p c^2 + m_{\bar{p}} c^2$, the particle–antiparticle pair can still be created for a short time. Indeed, Heisenberg's uncertainty principle in the (E, t) sector (as opposed to the usual (x, p) sector) means that $\Delta E \cdot \Delta t \sim \hbar$, meaning that for a short time $\Delta t \sim \hbar/\Delta E$ we can have an uncertainty in the energy ΔE, for instance such that $E + \Delta E > m_p c^2 + m_{\bar{p}} c^2$. This means that we can create a pair of *virtual particles*, that is particles which are forbidden by energy and momentum conservation to exist as asymptotic particles, but can exist as quantum fluctuations for a short time.
- Third, causality is violated by a single particle propagating via usual quantum mechanics formulas, even with the relativistic formula for the energy, $E = \sqrt{\vec{p}^2 + m^2}$.

The amplitude for propagation from $\vec{x}_0$ to $\vec{x}$ in a time t in quantum mechanics is

$$U(t) = \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle, \tag{1.1}$$

and replacing E, the eigenvalue of H, by $\sqrt{\vec{p}^2 + m^2}$, we obtain

$$U(t) = \langle \vec{x} | e^{-it\sqrt{\vec{p}^2+m^2}} | \vec{x}_0 \rangle = \frac{1}{(2\pi)^3} \int d^3\vec{p}\, e^{-it\sqrt{\vec{p}^2+m^2}} e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)}. \tag{1.2}$$

But

$$\begin{aligned} \int d^3\vec{p}\, e^{i\vec{p}\cdot\vec{x}} &= \int p^2 dp \int 2\pi \sin\theta d\theta\, e^{ipx\cos\theta} \\ &= \int p^2 dp \left[\frac{2\pi}{ipx}(e^{ipx} - e^{-ipx})\right] = \int p^2 dp \left[\frac{4\pi}{px}\sin(px)\right], \end{aligned} \tag{1.3}$$

and therefore

$$U(t) = \frac{1}{2\pi^2 |\vec{x} - \vec{x}_0|} \int_0^\infty p dp \sin(p|\vec{x} - \vec{x}_0|) e^{-it\sqrt{p^2+m^2}}. \tag{1.4}$$

For $x^2 \gg t^2$, we use a saddle-point approximation, which is the idea that the integral $I = \int dx e^{f(x)}$ can be approximated by the Gaussian around the saddle point (i.e. $I \simeq e^{f(x_0)} \int d\delta x e^{f''(x_0)\delta x^2} \simeq e^{f(x_0)} \sqrt{\pi/f''(x_0)}$, where x_0 is the saddle point) at whose position we have $f'(x_0) = 0$. Generally, if we are interested in leading behavior in some large parameter, the function $e^{f(x_0)}$ dominates $\sqrt{\pi/f''(x_0)}$ and we can just approximate $I \sim e^{f(x_0)}$.

In our case, we obtain

$$\frac{d}{dp}\left(ipx - it\sqrt{p^2 + m^2}\right) = 0 \Rightarrow x = \frac{tp}{\sqrt{p^2 + m^2}} \Rightarrow p = p_0 = \frac{imx}{\sqrt{x^2 - t^2}}. \tag{1.5}$$

Since we are at $x^2 \gg t^2$, we obtain

$$U(t) \propto e^{ip_0 x - it\sqrt{p_0^2 + m^2}} \sim e^{-\sqrt{x^2 - t^2}} \neq 0. \tag{1.6}$$

So we see that even much outside the lightcone, at $x^2 \gg t^2$, we have nonzero probability for propagation, meaning a breakdown of causality.

However, we will see that this problem is fixed in quantum field theory, which will be causal.

In quantum field theory, the fields describe many particles. One example of this fact that is easy to understand is the case of the electromagnetic field, $(\vec{E}, \vec{B}) \to F_{\mu\nu}$, which describes many photons. Indeed, we know from the correspondence principle of quantum mechanics that a classical state is equivalent to a state with many photons, and also that the number of photons is not a constant in any sense: we can define a (quantum) average number of photons that is related to the classical intensity of an electromagnetic beam, but the number of photons is not a classically measurable quantity.

We will describe processes involving many particles by *Feynman diagrams*, which will be an important part of this book. In quantum mechanics, a particle propagates forever, so its "Feynman diagram" is always a single line, as in Figure 1.1.

In quantum field theory, however, we will derive the mathematical form of Feynman diagrams, but the simple physical interpretation for which Feynman introduced them is that we can have processes where, for instance, a particle splits into two (or more) (see Figure 1.1(a)), two (or more) particles merge into one (see Figure 1.1(b)), or two (or more) particles of one type disappear and another type is created, like for instance in the annihilation of an e^+ (positron) with an e^- (electron) into a photon (γ) as in Figure 1.1(c), and so on.

Moreover, we can have (as we mentioned) virtual processes, like a photon γ creating an e^+e^- pair, which lives for a short time Δt and then annihilates into a γ, creating an e^+e^- virtual loop inside the propagating γ, as in Figure 1.2. Of course, $E, \vec{p}$ conservation means that $(E, \vec{p})$ is the same for the γ before and after the loop.

Next, we should review a few notions of classical field theory.

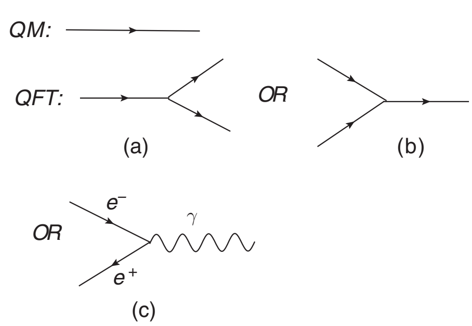

Figure 1.1 Quantum mechanics: particle goes on forever. Quantum field theory: particles can split (a), join (b), and particles of different types can appear and disappear, like in the quantum electrodynamics process (c).

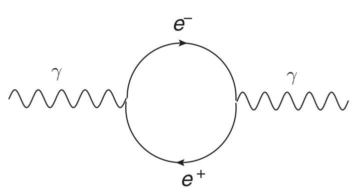

Figure 1.2 Virtual particles can appear for a short time in a loop. Here a photon creates a virtual electron–positron pair, which then annihilates back into the photon.

1.2 Classical Mechanics

Before doing that, however, we begin with an even quicker review of **classical mechanics**.

In classical mechanics, the description of a system is in terms of a Lagrangian $L(q_i, \dot{q}_i)$ for the variables $q_i(t)$, and the corresponding action

$$S = \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t)). \tag{1.7}$$

By varying the action with fixed boundary values for the variables $q_i(t)$ (i.e. $\delta S = 0$), we obtain the Euler–Lagrange equations (or equations of motion)

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{1.8}$$

We can also do a Legendre transformation from the Lagrangian $L(q_i, \dot{q}_i)$ to the Hamiltonian $H(q_i, p_i)$ in the usual way, by

$$H(p,q) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i), \tag{1.9}$$

where

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{1.10}$$

is the momentum canonically conjugate to the coordinate q_i.

Differentiating the Legendre transformation formula, we get the first-order Hamilton equations (instead of the second-order Lagrange equations)

$$\begin{aligned}\frac{\partial H}{\partial p_i} &= \dot{q}_i,\\ \frac{\partial H}{\partial q_i} &= -\frac{\partial L}{\partial q_i} = -\dot{p}_i.\end{aligned} \tag{1.11}$$

1.3 Classical Field Theory

The generalization of classical mechanics to **field theory** is obtained by considering instead of a set $\{q_i(t)\}_i$, which is a collection of given particles, fields $\phi(\vec{x}, t)$, where $\vec{x}$ is a generalization of i, and not a coordinate of a particle.

We will be interested in *local* field theories, which means all objects are integrals over $\vec{x}$ of functions defined at a point, in particular the Lagrangian is written as

$$L(t) = \int d^3\vec{x} \mathcal{L}(\vec{x}, t). \tag{1.12}$$

Here $\mathcal{L}$ is called the *Lagrange density*, but by an abuse of notation, one usually refers to it also as the Lagrangian.

We are also interested in *relativistic field theories*, which means that $\mathcal{L}(\vec{x}, t)$ is a relativistically invariant function of fields and their derivatives:

$$\mathcal{L}(\vec{x}, t) = \mathcal{L}(\phi(\vec{x}, t), \partial_\mu \phi(\vec{x}, t)). \tag{1.13}$$

Considering also several fields ϕ_a, we have an action written as

$$S = \int L dt = \int d^4 x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{1.14}$$

where $d^4 x = dt d^3 \vec{x}$ is the relativistically invariant volume element for spacetime.

The Euler–Lagrange equations are obtained in the same way, as

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right] = 0. \tag{1.15}$$

Note that one could think of $L(q_i)$ as a discretization over $\vec{x}$ of $\int d^3 \vec{x} \mathcal{L}(\phi_a)$, but that is not particularly useful.

In the Lagrangian we have relativistic fields, that is fields that have a well-defined transformation property under Lorentz transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.16}$$

namely

$$\phi'_i(x') = R_i{}^j \phi_i(x), \tag{1.17}$$

where i is some index for the fields, related to its Lorentz properties. We will come back to this later, but for now let us just observe that for a scalar field there is no i and $R \equiv 1$ (i.e. $\phi'(x') = \phi(x)$).

In this book I will use the convention for the spacetime metric with "mostly plus" on the diagonal, that is the Minkowski metric is

$$\eta_{\mu\nu} = diag(-1, +1, +1, +1). \tag{1.18}$$

Note that this is the convention that is the most natural in order to make heavy use of Euclidean field theory via Wick rotation, as we will do (by just redefining the time t by a factor of i), and so is very useful if we work with the functional formalism, where Euclidean field theory is essential.

On the contrary, for various reasons, people connected with phenomenology and making heavy use of the operator formalism often use the "mostly minus" metric ($\eta_{\mu\nu} = diag(+1, -1, -1, -1)$), for instance the standard textbook of Peskin and Schroeder [1] does so, so one has to be very careful when translating results from one convention to the other.

With this metric, the Lagrangian for a scalar field is generically

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) \\ &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\vec{\nabla}\phi|^2 - \frac{1}{2} m^2 \phi^2 - V(\phi),\end{aligned} \tag{1.19}$$

so is of the general type $\dot{q}^2/2 - \tilde{V}(q)$, as it should be (where the terms $1/2|\vec{\nabla}\phi|^2 + m^2\phi^2/2$ are also part of $\tilde{V}(q)$).

To go to the Hamiltonian formalism, we must first define the momentum canonically conjugate to the field $\phi(\vec{x})$ (remembering that $\vec{x}$ is a label like i):

$$p(\vec{x}) = \frac{\partial L}{\partial \dot{\phi}(\vec{x})} = \frac{\partial}{\partial \dot{\phi}(\vec{x})} \int d^3\vec{y} \mathcal{L}(\phi(\vec{y}), \partial_\mu \phi(\vec{y})) = \pi(\vec{x}) d^3\vec{x}, \tag{1.20}$$

where

$$\pi(\vec{x}) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(\vec{x})} \tag{1.21}$$

is a conjugate momentum density, but by an abuse of notation again will just be called conjugate momentum.

Then the Hamiltonian is

$$\begin{aligned} H &= \sum_{\vec{x}} p(\vec{x}) \dot{\phi}(\vec{x}) - L \\ &\to \int d^3\vec{x} [\pi(\vec{x}) \dot{\phi}(\vec{x}) - \mathcal{L}] \equiv \int d^3\vec{x} \mathcal{H}, \end{aligned} \tag{1.22}$$

where $\mathcal{H}$ is a Hamiltonian density.

1.4 Noether Theorem

The statement of the Noether theorem is that for every symmetry of the Lagrangian L, there is a corresponding conserved charge.

The best known examples are the time translation $t \to t + a$, corresponding to conserved energy E, and the space translation $\vec{x} \to \vec{x} + \vec{a}$, corresponding to conserved momentum $\vec{p}$, together making the spacetime translation $x^\mu \to x^\mu + a^\mu$, corresponding to conserved 4-momentum P^μ. The *Noether currents* corresponding to these charges form the energy–momentum tensor $T_{\mu\nu}$.

Consider the symmetry $\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta \phi$ that transforms the Lagrangian density as

$$\mathcal{L} \to \mathcal{L} + \alpha \partial_\mu J^\mu, \tag{1.23}$$

such that the action $S = \int d^4x \mathcal{L}$ is invariant, if the fields vanish on the boundary, usually considered at $t = \pm\infty$, since the boundary term

$$\int d^4x \partial_\mu J^\mu = \oint_{bd} dS_\mu J^\mu = \int d^3\vec{x} J^0 |_{t=-\infty}^{t=+\infty} \tag{1.24}$$

is then zero. In this case, there exists a conserved current j^μ, that is

$$\partial_\mu j^\mu(x) = 0, \tag{1.25}$$

where

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu. \tag{1.26}$$

For linear symmetries (symmetry transformations linear in ϕ), we can define

$$(\alpha \Delta\phi)^i \equiv \alpha^a (T^a)^i{}_j \phi^j \tag{1.27}$$

such that, if $J^\mu = 0$, we have the Noether current

$$j^{\mu,a} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (T^a)^i{}_j \phi^j. \tag{1.28}$$

Applying this general formalism to translations, $x^\mu \to x^\mu + a^\mu$, we obtain, for an infinitesimal parameter a^μ:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi, \tag{1.29}$$

which are the first terms in the Taylor expansion around x. The corresponding conserved current is therefore

$$T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L}\delta^\mu_\nu, \tag{1.30}$$

where we have added a term $J^\mu_{(\nu)} = \mathcal{L}\delta^\mu_\nu$ to get the conventional definition of the *energy–momentum tensor* or *stress–energy tensor*. The conserved charges are integrals of the energy–momentum tensor (i.e. P^μ). Note that the above translation can be considered as also giving the term $J^\mu_{(\nu)}$ from the general formalism, since we can check that for $\alpha^\nu = a^\nu$, the Lagrangian changes by $\partial_\mu J^\mu_{(\nu)}$.

1.5 Fields and Lorentz Representations

The Lorentz group is $SO(1,3)$, that is an orthogonal group that generalizes $SO(3)$, the group of rotations in the (Euclidean) three spatial dimensions.

Its basic objects in the fundamental representation, defined as the representation that acts on coordinates x^μ (or rather dx^μ), are called $\Lambda^\mu{}_\nu$, and thus

$$dx'^\mu = \Lambda^\mu{}_\nu dx^\nu. \tag{1.31}$$

If η is the matrix $\eta_{\mu\nu}$, the Minkowski metric $diag(-1,+1,+1,+1)$, the orthogonal group $SO(1,3)$ is the group of elements Λ that satisfy

$$\Lambda \eta \Lambda^T = \eta. \tag{1.32}$$

Note that the usual rotation group $SO(3)$ is an orthogonal group satisfying

$$\Lambda\Lambda^T = \mathbf{1} \Rightarrow \Lambda^{-1} = \Lambda^T, \tag{1.33}$$

but we should actually write this as

$$\Lambda \mathbf{1} \Lambda^T = \mathbf{1}, \tag{1.34}$$

which admits a generalization to $SO(p, q)$ groups as

$$\Lambda g \Lambda^T = g, \tag{1.35}$$

where $g = diag(-1, \ldots, -1, +1, \ldots, +1)$ with p minuses and q pluses. In the above, Λ satisfies the group property, namely if Λ_1, Λ_2 belong in the group, then

$$\Lambda_1 \cdot \Lambda_2 \equiv \Lambda \tag{1.36}$$

is also in the group.

General representations are a generalization of (1.31), namely instead of acting on x, the group acts on a vector space ϕ^a by

$$\phi'^a(\Lambda x) = R(\Lambda)^a{}_b \phi^b(x), \tag{1.37}$$

such that it respects the group property, that is

$$R(\Lambda_1)R(\Lambda_2) = R(\Lambda_1 \cdot \Lambda_2). \tag{1.38}$$

Group elements are represented for infinitesimally small parameters β^a as exponentials of the *Lie algebra generators* in the R representation $t_a^{(R)}$, that is

$$R(\beta) = e^{i\beta^a t_a^{(R)}}. \tag{1.39}$$

The statement that $t_a^{(R)}$ form a Lie algebra is the statement that we have a relation

$$[t_a^{(R)}, t_b^{(R)}] = i f_{ab}{}^c t_c^{(R)} \tag{1.40}$$

where $f_{ab}{}^c$ are called the *structure constants*. Note that the factor of i is conventional, with this definition we can have Hermitian generators, for which $\mathrm{Tr}(t_a t_b) = \delta_{ab}$; if we redefine t_a by an i we can remove it from there, but then $\mathrm{Tr}(t_a t_b)$ can be put only to $-\delta_{ab}$ (anti-Hermitian generators).

The representations of the Lorentz group are:

- Bosonic. Scalars ϕ for which $\phi'(x') = \phi(x)$; vectors like the electromagnetic field $A_\mu = (\phi, \vec{A})$ that transform as ∂_μ (covariant) or dx^μ (contravariant), and representations which have products of indices, like for instance the electromagnetic field strength $F_{\mu\nu}$ which transforms as

$$F'_{\mu\nu}(\Lambda x) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}(x), \tag{1.41}$$

  where $\Lambda_\mu{}^\nu = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma$. For fields with more indices, $B^{\nu_1 \ldots \nu_j}_{\mu_1 \ldots \mu_k}$, it transforms as the appropriate products of Λ.
- Fermionic. Spinors, which will be treated in more detail later on in the book. For now, let us just say that fundamental spinor representations ψ are acted upon by gamma matrices γ^μ.

The Lie algebra of the Lorentz group $SO(1, 3)$ is

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\eta_{\mu\rho}J_{\nu\sigma} + i\eta_{\mu\sigma}J_{\nu\rho} - i\eta_{\nu\sigma}J_{\mu\rho} + i\eta_{\nu\rho}J_{\mu\sigma}. \tag{1.42}$$

Note that if we denote $a \equiv (\mu\nu)$, $b \equiv (\rho\sigma)$, and $c \equiv (\lambda\pi)$, we then have

$$f_{ab}{}^{c} = -\eta_{\mu\rho}\delta^{\lambda}_{[\nu}\delta^{\pi}_{\sigma]} + \eta_{\mu\sigma}\delta^{\lambda}_{[\nu}\delta^{\pi}_{\rho]} - \eta_{\nu\sigma}\delta^{\lambda}_{[\mu}\delta^{\pi}_{\rho]} + \eta_{\nu\rho}\delta^{\lambda}_{[\mu}\delta^{\pi}_{\sigma]}, \tag{1.43}$$

so (1.42) is indeed of the Lie algebra type.

The Lie algebra $SO(1,3)$ is (modulo some global subtleties) the same as the product of two $SU(2)$s (i.e. $SU(2) \times SU(2)$), which can be seen by first defining

$$J_{0i} \equiv K_i; \quad J_{ij} \equiv \epsilon_{ijk}J_k, \tag{1.44}$$

where $i, j, k = 1, 2, 3$, and then redefining

$$M_i \equiv \frac{J_i + iK_i}{2}; \quad N_i \equiv \frac{J_i - iK_i}{2}, \tag{1.45}$$

after which we obtain

$$\begin{aligned} [M_i, M_j] &= i\epsilon_{ijk}M_k, \\ [N_i, N_j] &= i\epsilon_{ijk}N_k, \\ [M_i, N_j] &= 0, \end{aligned} \tag{1.46}$$

which we leave as an exercise to prove.

Important Concepts to Remember

- Quantum field theory is a relativistic quantum mechanics, which necessarily describes an arbitrary number of particles.
- Particle–antiparticle pairs can be created and disappear, both as real (energetically allowed) and virtual (energetically disallowed, only possible due to Heisenberg's uncertainty principle).
- If we use the usual quantum mechanics rules, even with $E = \sqrt{p^2 + m^2}$, we have causality breakdown: the amplitude for propagation is nonzero even much outside the lightcone.
- Feynman diagrams represent the interaction processes of creation and annihilation of particles.
- When generalizing classical mechanics to field theory, the label i is generalized to $\vec{x}$ in $\phi(\vec{x}, t)$, and we have a Lagrangian density $\mathcal{L}(\vec{x}, t)$, conjugate momentum density $\pi(\vec{x}, t)$, and Hamiltonian density $\mathcal{H}(\vec{x}, t)$.
- For relativistic and local theories, $\mathcal{L}$ is a relativistically invariant function defined at a point x^μ.
- The Noether theorem associates a conserved current ($\partial_\mu j^\mu = 0$) with a symmetry of the Lagrangian L, in particular the energy–momentum tensor T^μ_ν with translations $x^\mu \to x^\mu + a^\mu$.
- Lorentz representations act on the fields ϕ^a, and are the exponentials of Lie algebra generators.
- The Lie algebra of $SO(1,3)$ splits into two $SU(2)$s.

Further Reading

See, for instance, sections 2.1 and 2.2 in [1] and chapter 1 in [2].

Exercises

1. Prove that for the Lie algebra of the Lorentz group

$$[J_{\mu\nu}, J_{\rho\sigma}] = -\left(-i\eta_{\mu\rho}J_{\nu\sigma} + i\eta_{\mu\sigma}J_{\nu\rho} - i\eta_{\nu\sigma}J_{\mu\rho} + i\eta_{\nu\rho}J_{\mu\sigma}\right), \tag{1.47}$$

if we define

$$J_{0i} \equiv K_i; \quad J_{ij} \equiv \epsilon_{ijk}J_k,$$
$$M_i \equiv \frac{J_i + iK_i}{2}; \quad N_i \equiv \frac{J_i - iK_i}{2}, \tag{1.48}$$

we obtain that the M_i and N_i satisfy

$$[M_i, M_j] = i\epsilon_{ijk}M_k,$$
$$[N_i, N_j] = i\epsilon_{ijk}N_k,$$
$$[M_i, N_j] = 0. \tag{1.49}$$

2. Consider the action in Minkowski space

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}(\not{D} + m)\psi - (D_\mu\phi)^* D^\mu\phi\right), \tag{1.50}$$

where $D_\mu = \partial_\mu - ieA_\mu$, $\not{D} = D_\mu\gamma^\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\bar{\psi} = \psi^\dagger i\gamma_0$, ψ is a spinor field and ϕ is a scalar field, and γ_μ are the gamma matrices, satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Consider the electromagnetic $U(1)$ transformation

$$\psi'(x) = e^{ie\lambda(x)}\psi(x); \quad \phi'(x) = e^{ie\lambda(x)}\phi(x); \quad A'_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x). \tag{1.51}$$

Calculate the Noether current.

3. Find the invariances of the model for N real scalars Φ^I, with Lagrangian

$$\mathcal{L} = g_{IJ}(\Phi^I\Phi^I)\partial_\mu\Phi^I\partial^\mu\Phi^I \tag{1.52}$$

in the case of a general metric g_{IJ}, and in the particular case of $g_{IJ} = \eta_{IJ}$ (at least in a local neighborhood in scalar space).

4. Calculate the equations of motion of the Dirac–Born–Infeld (DBI) scalar Lagrangian

$$\mathcal{L} = -\frac{1}{L^4}\sqrt{1 + L^4[g(\phi)(\partial_\mu\phi)^2 + m^2\phi^2]}. \tag{1.53}$$