Game Theory
Second Edition

Covering both noncooperative and cooperative games, this comprehensive introduction to game theory also includes some advanced chapters on auctions, games with incomplete information, games with vector payoffs, stable matchings, and the bargaining set. Mathematically oriented, the book presents every theorem alongside a proof. The material is presented clearly and every concept is illustrated with concrete examples from a broad range of disciplines. With numerous exercises the book is a thorough and extensive guide to game theory from undergraduate through graduate courses in economics, mathematics, computer science, engineering, and life sciences to being an authoritative reference for researchers.

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Game Theory
Second Edition

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To Michael Maschler
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The book makes use of a large number of notations; we have striven to stick to accepted notations and to be consistent throughout the book. The coordinates of a vector are always denoted by a subscript index, \( x = (x_i)_{i=1}^n \), while the indices of the elements of sequences are always denoted by a superscript index, \( x^1, x^2, \ldots \). The index of a player in a set of players is always denoted by a subscript index, while a time index (in repeated games) is always denoted by a superscript index. The end of the proof of a theorem is indicated by \( \Box \), the end of an example is indicated by \( \diamond \), and the end of a remark is indicated by \( \blacklozenge \).

For convenience we provide a list of the mathematical notations used throughout the book, accompanied by a short explanation and the pages on which they are formally defined. The notations that appear below are those that are used more than once.

- \( 0 \) chance move in an extensive-form game 50
- \( \emptyset \) origin of a Euclidean space 579
- \( \emptyset \) strategy used by a player who has no decision vertices in an extensive-form game 5
- \( 1_A \) function that is equal to 1 on event \( A \) and to 0 otherwise 595
- \( 2^Y \) collection of all subsets of \( Y \) 336
- \( |X| \) number of elements in finite set \( X \) 603
- \( ||x||_\infty \) \( L_\infty \) norm, \( ||x||_\infty := \max_{i=1,2,\ldots,n} |x_i| \) 539
- \( ||x|| \) norm of a vector, \( ||x|| := \sqrt{\sum_{i=1}^d (x_i)^2} \) 579
- \( A \lor B \) maximum matching (for men) in a matching problem 942
- \( A \land B \) maximum matching (for women) in a matching problem 943
- \( A \subseteq B \) set \( A \) contains set \( B \) or is equal to it 942
- \( A \subset B \) set \( A \) strictly contains set \( B \) 942
- \( \langle x, y \rangle \) inner product 579
- \( \langle x^0, \ldots, x^k \rangle \) \( k \)-dimensional simplex 965
- \( \asymp_i \) preference relation of player \( i \) 13
- \( >_i \) strict preference relation of player \( i \) 10
- \( \approx_i \) indifference relation of player \( i \) 10, 944
- \( \asymp_p \) preference relation of an individual 905
- \( >Q \) strict preference relation of society 905
- \( \approx Q \) indifference relation of society 905
- \( x \geq y \) \( x_k \geq y_k \) for each coordinate \( k \), where \( x, y \) are vectors in a Euclidean space 676
- \( x > y \) \( x \geq y \) and \( x \neq y \) 676
### Notations

- \( x \succ y \): \( x_k > y_k \) for each coordinate \( k \), where \( x, y \) are vectors in a Euclidean space
- \( x + y \): sum of vectors in a Euclidean space, \( (x + y)_k := x_k + y_k \)
- \( xy \): coordinatewise product of vectors in a Euclidean space, \( (xy)_k := x_k y_k \)
- \( x + S \): \( \{ x + s : s \in S \} \), where \( x \in \mathbb{R}^d \) and \( S \subseteq \mathbb{R}^d \)
- \( xS \): \( \{ xs : s \in S \} \), where \( x \in \mathbb{R}^d \) and \( S \subseteq \mathbb{R}^d \)
- \( cx \): product of real number \( c \) and vector \( x \)
- \( cS \): \( \{ cs : s \in S \} \), where \( c \) is a real number and \( S \subseteq \mathbb{R}^d \)
- \( S + T \): sum of sets, \( S + T := \{ x + y : x \in S, y \in T \} \)
- \([c] \): smallest integer greater than or equal to \( c \)
- \([c] \): largest integer less than or equal to \( c \)
- \( x^T \): transpose of a vector, column vector that corresponds to row vector \( x \)
- \( \text{argmax}_{x \in X} f(x) \): set of all \( x \) where function \( f \) attains its maximum in the set \( X \)
- \( a(i) \): producer \( i \)'s initial endowment in a market
- \( A \): set of actions in a decision problem with experts
- \( A \): set of alternatives
- \( A_i \): player \( i \)'s action set in an extensive-form game, \( A_i := \bigcup_{j=1}^k A(U_j^i) \)
- \( A_k \): possible outcome of a game
- \( A(x) \): set of available actions at vertex \( x \) in an extensive-form game
- \( A(U_i) \): set of available actions at information set \( U_i \) of player \( i \) in an extensive-form game
- \( b_i \): buyer \( i \)'s bid in an auction
- \( b(S) \): \( b(S) = \sum_{b \in S} b_i \) where \( b \in \mathbb{R}^N \)
- \( \text{br}_1(y) \): Player I's set of best replies to strategy \( y \)
- \( \text{br}_2(x) \): Player II's set of best replies to strategy \( x \)
- \( B_i \): player \( i \)'s belief operator
- \( B_i^\omega \): set of states of the world in which the probability that player \( i \) ascribes to event \( E \) is at least \( p \), \( B_i^\omega(E) := \{ \omega \in Y : \pi_i(E \mid \omega) \geq p \} \)
- \( \text{BZ}_i(N; v) \): Banzhaf value of a coalitional game
- \( B \): coalitional structure
- \( B_i^T \): set of behavior strategies of player \( i \) in a \( T \)-repeated game
- \( B_i^\infty \): set of behavior strategies of player \( i \) in an infinitely repeated game
- \( c \): coalitional function of a cost game
- \( c_+ \): maximum of \( c \) and 0
- \( c_i \): \( c_i(v_i) := v_i - \frac{1 - E_i(v_i)}{f_i(v_i)} \)
- \( C \): function that dictates the amount that each buyer pays given the vector of bids in an auction
### Notations

- **$C(x)$** set of children of vertex $x$ in an extensive-form game
- **$C(N, v)$** core of a coalitional game
- **$C(N, v; \mathcal{B})$** core for a coalitional structure
- **conv$\{x_1, \ldots, x_k\}$** smallest convex set that contains the vectors $\{x_1, \ldots, x_k\}$, also called the convex hull of $\{x_1, \ldots, x_k\}$
- **$d$** disagreement point of a bargaining game
- **$d_i$** debt to creditor $i$ in a bankruptcy problem
- **$d(x,y)$** Euclidean distance between two vectors in Euclidean space
- **$d(x,S)$** Euclidean distance between point and set
- **$D(\alpha, x)$** collection of coalitions whose excess is at least $\alpha$
- **$e(S,x)$** excess of coalition $S$, $e(S,x) := v(S) - x(S)$
- **$E$** set of feasible payoffs in a repeated game
- **$E$** social welfare function
- **$F_i$** cumulative distribution function of buyer $i$’s private values in an auction
- **$F_i(\omega)$** atom of the partition $\mathcal{F}_i$ that contains $\omega$
- **$F^N$** cumulative distribution function of joint distribution of vector of private values in an auction
- **$\mathcal{F}$** collection of all subgames in the game of chess
- **$\mathcal{F}_i$** player $i$’s information in an Aumann model of incomplete information
- **$g^T$** average payoff up to stage $T$ (including) in a repeated game
- **$G$** graph
- **$G$** social choice function
- **$h$** history of a repeated game
- **$h_t$** history at stage $t$ of a repeated game
- **$H(t)$** set of $t$-stage histories of a repeated game
- **$H(\infty)$** set of plays in an infinitely repeated game
- **$H(\alpha, \beta)$** hyperplane
- **$H^+(\alpha, \beta)$** half-space
- **$H^-(\alpha, \beta)$** half-space
- **$i$** player
- **$\neg i$** set of all players except player $i$
## Notations

\( I \)  
function that dictates the winner of an auction given the vector of bids 474

\( J \)  
number of lotteries that compose a compound lottery 14

\( J(x) \)  
player who chooses a move at vertex \( x \) of an extensive-form game 44

\(-k\)  
player who is not \( k \) in a two-player game 580

\( k_i \)  
number of information sets of player \( i \) in an extensive-form game 54

\( K \)  
number of outcomes of a game 16

\( K_i \)  
player \( i \)'s knowledge operator 336

\( KS, KS(S) \)  
Kalai–Smorodinsky solution to bargaining games 699

\( L \)  
lottery: \( L = [p_1(A_1), p_2(A_2), \ldots, p_K(A_K)] \) 13

\( L \)  
number of commodities in a market 751

\( \hat{L} \)  
compound lottery: \( \hat{L} = [q_1(L_1), \ldots, q_J(L_J)] \) 14

\( \mathcal{L} \)  
set of lotteries 13

\( \hat{\mathcal{L}} \)  
set of compound lotteries 15

\( m(\epsilon) \)  
minimal coordinate of vector \( \epsilon \) 275, 279

\( m_i \)  
number of pure strategies of player \( i \) 146

\( m_i(S) \)  
highest possible payoff to player \( i \) in a bargaining game 694

\( M \)  
maximal absolute value of a payoff in a game 528

\( M_{m,l} \)  
space of matrices of dimension \( m \times l \) 213

\( M(\epsilon) \)  
maximal coordinate of vector \( \epsilon \) 275, 279

\( \mathcal{M}(N; v; B) \)  
bargaining set for coalitional structure \( B \) 834

\( n \)  
number of players 77

\( n \)  
number of buyers in an auction 474

\( n_x \)  
number of vertices in subgame \( \Gamma(x) \) 5

\( N \)  
set of players 43, 881, 710

\( N \)  
set of buyers in an auction 474

\( N \)  
set of individuals 904

\( N \)  
set of producers in a market 751

\( \mathbb{N} \)  
set of natural numbers, \( \mathbb{N} := \{1, 2, 3, \ldots\} \)

\( \mathcal{N} \)  
\( \mathcal{N}(S, d) \), Nash’s solution to bargaining games 681

\( \mathcal{N}(N; v) \)  
nucleolus of a coalitional game 853

\( \mathcal{N}(N; v; B) \)  
nucleolus of a coalitional game for coalitional structure \( B \) 853

\( \mathcal{N}(N; v; K) \)  
nucleolus relative to set \( K \) 852

\( O \)  
set of outcomes 13, 43

\( p \)  
common prior in a Harsanyi game with incomplete information 358

\( pk \)  
probability that the outcome of lottery \( L \) is \( A_k \) 13

\( px \)  
probability distribution over actions at chance move \( x \) 50

\( P \)  
binary relation 905
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<td>$P$</td>
<td>common prior in an Aumann model of incomplete information</td>
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<td>$P_\sigma(x)$</td>
<td>probability that the play reaches vertex $x$ when the players implement strategy vector $\sigma$ in an extensive-form game</td>
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<tr>
<td>$P_\sigma(U)$</td>
<td>probability that the play reaches a vertex in information set $U$ when the players implement strategy vector $\sigma$ in an extensive-form game</td>
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<td>$p^N$</td>
<td>vector of preference relations</td>
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<td>$PO(S)$</td>
<td>set of efficient (Pareto optimal) points in $S$</td>
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<tr>
<td>$PO^W(S)$</td>
<td>set of weakly efficient points in $S$</td>
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<tr>
<td>$P(A)$</td>
<td>set of all strict preference relations over a set of alternatives $A$</td>
</tr>
<tr>
<td>$P(N)$</td>
<td>collection of nonempty subsets of $N$, $P(N) := {S \subseteq N, S \neq \emptyset}$</td>
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<tr>
<td>$P^*(A)$</td>
<td>set of all preference relations over a set of alternatives $A$</td>
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<td>$P^\mathcal{N}(N; v)$</td>
<td>prenucleolus of a coalitional game</td>
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<tr>
<td>$P^\mathcal{N}(N; v; \mathcal{B})$</td>
<td>prenucleolus of a coalitional game for coalitional structure $\mathcal{B}$</td>
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<td>$q$</td>
<td>quota in a weighted majority game</td>
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<td>$q(w)$</td>
<td>minimal weight of a winning coalition in a weighted majority game, $q(w) := \min_{S \in \mathcal{W}^m} w(S)$</td>
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<tr>
<td>$Q_{++}$</td>
<td>set of positive rational numbers</td>
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<td>$r_k$</td>
<td>total probability that the result of a compound lottery is $A_k$</td>
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<td>$R_1(p)$</td>
<td>set of possible payoffs when Player 1 plays mixed action $p$, $R_1(p) := {pu^q : q \in \Delta(J)}$</td>
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<tr>
<td>$R_2(p)$</td>
<td>set of possible payoffs when Player 2 plays mixed action $q$, $R_2(p) := {pu^q : q \in \Delta(I)}$</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>real line</td>
</tr>
<tr>
<td>$\mathbb{R}_+$</td>
<td>set of nonnegative numbers</td>
</tr>
<tr>
<td>$\mathbb{R}_{++}$</td>
<td>set of positive numbers</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>$n$-dimensional Euclidean space</td>
</tr>
<tr>
<td>$\mathbb{R}^+_n$</td>
<td>nonnegative orthant in an $n$-dimensional Euclidean space, $\mathbb{R}^+_n := {x \in \mathbb{R}^n : x_i \geq 0, \ \forall i = 1, 2, \ldots, n}$</td>
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<td>$\mathbb{R}^S$</td>
<td>$</td>
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<td>$\text{range}(G)$</td>
<td>range of a social choice function</td>
</tr>
<tr>
<td>$s$</td>
<td>strategy vector</td>
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<tr>
<td>$s$</td>
<td>function that assigns a state of nature to each state of the world</td>
</tr>
<tr>
<td>$s'$</td>
<td>action vector played at stage $t$ of a repeated game</td>
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<tr>
<td>$s_i$</td>
<td>strategy of player $i$</td>
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<td>state of nature that corresponds to type vector $t$ in a Harsanyi game with incomplete information</td>
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<tr>
<td>$s^{-1}(C)$</td>
<td>set of states of the world that correspond to a state of nature in $C$, $s^{-1}(C) := {\omega \in Y: s(\omega) \in C}$</td>
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<tr>
<td>$S$</td>
<td>set of all vectors of pure strategies</td>
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<tr>
<td>$S$</td>
<td>set of states of nature in models of incomplete information</td>
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<tr>
<td>$S$</td>
<td>set of states of nature in a decision problem with experts</td>
</tr>
<tr>
<td>$S_i$</td>
<td>set of player $i$'s pure strategies</td>
</tr>
<tr>
<td>$\text{supp}$</td>
<td>support of a probability distribution</td>
</tr>
<tr>
<td>$\text{supp}$</td>
<td>support of a vector in $\mathbb{R}^n$</td>
</tr>
<tr>
<td>$t_i$</td>
<td>player $i$'s type in models of incomplete information</td>
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<tr>
<td>$T$</td>
<td>set of vectors of types in a Harsanyi model of incomplete information</td>
</tr>
<tr>
<td>$T$</td>
<td>number of stages in a finitely repeated game</td>
</tr>
<tr>
<td>$T_i$</td>
<td>player $i$'s type set in a Harsanyi model of incomplete information</td>
</tr>
<tr>
<td>$u$</td>
<td>payoff function in a strategic-form game</td>
</tr>
<tr>
<td>$u_i$</td>
<td>player $i$'s utility function</td>
</tr>
<tr>
<td>$u_i$</td>
<td>player $i$'s payoff function</td>
</tr>
<tr>
<td>$u_i^p$</td>
<td>producer $i$'s production function in a market</td>
</tr>
<tr>
<td>$u^t_i$</td>
<td>payoff of player $i$ at stage $t$ in a repeated game</td>
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<tr>
<td>$u^t$</td>
<td>vector of payoffs at stage $t$ in a repeated game</td>
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<tr>
<td>$u(s)$</td>
<td>outcome of a game under strategy vector $s$</td>
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<tr>
<td>$U_i^j$</td>
<td>information set of player $i$ in an extensive-form game</td>
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<tr>
<td>$U_i$</td>
<td>mixed extension of player $i$'s payoff function</td>
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<tr>
<td>$U(C)$</td>
<td>uniform distribution over set $C$</td>
</tr>
<tr>
<td>$U[\alpha]$</td>
<td>scalar payoff function generated by projecting the payoffs in direction $\alpha$ in a game with payoff vectors</td>
</tr>
<tr>
<td>$v$</td>
<td>value of a two-player zero-sum game</td>
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<tr>
<td>$v$</td>
<td>coalitional function of a coalitional game</td>
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<tr>
<td>$\bar{v}$</td>
<td>maxmin value of a two-player non-zero-sum game</td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>minmax value of a two-player non-zero-sum game</td>
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<tr>
<td>$\bar{v}$</td>
<td>maximal private value of buyers in an auction</td>
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<tr>
<td>$v_0$</td>
<td>root of a game tree</td>
</tr>
<tr>
<td>$v_i$</td>
<td>buyer $i$'s private value in an auction</td>
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<tr>
<td>$v^s$</td>
<td>superadditive closure of a coalitional game</td>
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<tr>
<td>$\bar{v}_i$</td>
<td>player $i$'s maxmin value in a strategic-form game</td>
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<tr>
<td>$\bar{v}_i$</td>
<td>player $i$'s minmax value in a strategic-form game</td>
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<tr>
<td>$\text{val}(A)$</td>
<td>value of a two-player zero-sum game whose payoff function is given by matrix $A$</td>
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<tr>
<td>$V$</td>
<td>set of edges in a graph</td>
</tr>
</tbody>
</table>
Notations

\( \set V \) \hspace{1cm} set of individually rational payoffs in a repeated game \hspace{1cm} 538

\( \set V_0 \) \hspace{1cm} set of vertices in an extensive-form game where a chance move takes place \hspace{1cm} 43

\( \set V_i \) \hspace{1cm} set of player \( i \)'s decision points in an extensive-form game \hspace{1cm} 43

\( \set V_i \) \hspace{1cm} random variable representing buyer \( i \)'s private value in an auction \hspace{1cm} 475

\( \set V \) \hspace{1cm} buyer’s set of possible private values in a symmetric auction \hspace{1cm} 479

\( \set V_j \) \hspace{1cm} buyer \( i \)'s set of possible private values \hspace{1cm} 474

\( \set V^N \) \hspace{1cm} set of vectors of possible private values: \( \set V^N := \set V_1 \times \set V_2 \times \cdots \times \set V_n \) \hspace{1cm} 474

\( w_i \) \hspace{1cm} player \( i \)'s weight in a weighted majority game \hspace{1cm} 714

\( \set W \) \hspace{1cm} collection of minimal winning coalitions in a simple monotonic game \hspace{1cm} 873

\( x_{-i} \) \hspace{1cm} \( x_{-i} := (x_j)_{j \neq i} \) \hspace{1cm} 85

\( x(S) \) \hspace{1cm} \( x(S) := \sum_{i \in X} x_i \), where \( x \in \mathbb{R}^N \) \hspace{1cm} 719

\( X \) \hspace{1cm} \( X := \times_{i \in X} X_i \) \hspace{1cm} 2

\( X_k \) \hspace{1cm} space of belief hierarchies of order \( k \) \hspace{1cm} 451

\( X_{-i} \) \hspace{1cm} \( X_{-i} := \times_{j \neq i} X_j \) \hspace{1cm} 85

\( X(n) \) \hspace{1cm} standard \( (n-1) \)-dimensional simplex, \( X(n) := \{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, \ x_i \geq 0 \ \forall i \} \) \hspace{1cm} 981

\( X(N; v) \) \hspace{1cm} set of imputations in a coagional game, \( X(N; v) := \{ x \in \mathbb{R}^n : x(N) = v(N), x_i \geq v(i) \ \forall i \in N \} \) \hspace{1cm} 724, 850

\( X^0(N; v) \) \hspace{1cm} set of preimputations, \( X^0(N; v) := \{ x \in \mathbb{R}^n : x(N) = v(N) \} \) \hspace{1cm} 852

\( X(B; v) \) \hspace{1cm} set of imputations for coalitional structure \( B \), \( X(B; v) := \{ x \in \mathbb{R}^n : x(S) = v(S) \ \forall S \in B, x_i \geq v_i \ \forall i \} \) \hspace{1cm} 724

\( X^0(B; v) \) \hspace{1cm} set of preimputations for coalitional structure \( B \), \( X^0(B; v) := \{ x \in \mathbb{R}^n : x(S) = v(S) \ \forall S \in B \} \) \hspace{1cm} 852

\( Y \) \hspace{1cm} set of states of the world \hspace{1cm} 334, 345

\( \bar{Y}(\omega) \) \hspace{1cm} minimal belief subspace in state of the world \( \omega \) \hspace{1cm} 411

\( \bar{Y}_i(\omega) \) \hspace{1cm} minimal belief subspace of player \( i \) in state of the world \( \omega \) \hspace{1cm} 412

\( Z_k \) \hspace{1cm} space of coherent belief hierarchies of order \( k \) \hspace{1cm} 454

\( Z(P, Q; R) \) \hspace{1cm} preference relation in which alternatives in \( R \) are preferred to alternatives not in \( R \), the preference over alternatives in \( R \) is determined by \( P \), and the preference over alternatives not in \( R \) is determined by \( Q \) \hspace{1cm} 914

\( Z(P^N, Q^N; R) \) \hspace{1cm} preference profile in which the preference of individual \( i \) is \( Z(P_i, Q_i; R) \) \hspace{1cm} 914

\( \beta_i \) \hspace{1cm} buyer \( i \)'s strategy in an auction \hspace{1cm} 475

\( \beta_i \) \hspace{1cm} buyer \( i \)'s strategy in a selling mechanism \hspace{1cm} 502

\( \beta_i^w \) \hspace{1cm} buyer \( i \)'s strategy in a direct selling mechanism in which he reports his private value \hspace{1cm} 503

\( \Gamma \) \hspace{1cm} extensive-form game \hspace{1cm} 43, 50, 54
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<td>extension of a strategic-form game to mixed strategies</td>
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<td>Γₜ</td>
<td>T-stage repeated game</td>
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<td>discounted game with discount factor λ</td>
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<td>Γ(ᵣ)</td>
<td>subgame of an extensive-form game that starts at vertex r</td>
<td>5, 45, 55</td>
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<td>Γ⁺(p)</td>
<td>extended game that includes a chance move that selects</td>
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<td></td>
<td>a vector of recommendations according to the probability</td>
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<td>distribution p in the definition of a correlated equilibrium</td>
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<td>Δ(S)</td>
<td>set of probability distributions over S</td>
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<td>ε</td>
<td>vector of constraints in the definition of perfect equilibrium</td>
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<td>strategy of the player who is not player k in a two-player game</td>
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<td>τᵢₜₖ</td>
<td>strategy in a game with an outside observer in which</td>
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