# 1
# Examples

In this introductory chapter, we list a number of concrete examples of $\tau$-functions, noting the elements they have in common, but postponing a formal definition to subsequent chapters.

The first case is the simplest nonlinear periodic Hamiltonian system with one degree of freedom: the pendulum. In the Hamilton–Jacobi approach, Hamilton's characteristic function, evaluated on the energy level sets, is the logarithmic derivative of the Weierstrass $\sigma$-function. This is our first example of a $\tau$-function. The equations of motion are expressible as a bilinear equation for the $\sigma$-function, providing the first instance of an equation of Hirota type.

Turning to nonlinear integrable evolution equations that are PDE's in one spatial and one time dimension, such as the KdV equation, the simplest reduction is to travelling wave solutions with constant velocity. These again satisfy a Weierstrass-type equation, just like the pendulum. The separatrix, where the $\tau$-function is simply a hyperbolic cosine, corresponds to 1-soliton solutions. The $\tau$-functions corresponding to multisoliton solutions are expressed as the determinant of a matrix whose entries are linear exponential functions of the flow variables, which again satisfies a bilinear system of Hirota type. Multisoliton solutions of the more general integrable KP (Kadomtsev–Petviashvili) hierarchy are similarly given in terms of $\tau$-functions having determinantal exponential form, also satisfying the Hirota bilinear equations.

We next consider the basic building blocks from which all KP $\tau$-functions are constructed: the Schur functions, which are polynomials in the flow parameters, whose logarithmic derivatives provide rational solutions of the hierarchy. Other examples include the Toda lattice, an integrable multiparticle system on the line with exponential nearest-neighbour interactions and the Calogero–Moser system, another integrable multiparticle system on the line whose dynamics coincide with the pole dynamics of rational solutions of the KP hierarchy. The "ultimate" generalization of the pendulum then follows: the so-called *finite gap* or multi-quasi-periodic solutions of the KP hierarchy, where the $\tau$-function is simply expressible in term of multivariable Riemann $\theta$ functions associated to the period lattice of an algebraic curve of arbitrary genus.

Further specific examples of KP $\tau$-functions are provided by the partition functions for various types of random matrix models. These include cases where the matrix integrals do not necessarily converge, nor does the expansion in the basis of Schur functions. They may, however, be viewed as formal expansions that serve as generating functions for various combinatorial invariants such as: intersection indices on the moduli space of marked Riemann surfaces, or Hurwitz numbers, which enumerate branched covers of the Riemann sphere.

The characteristic features shared by all these examples are listed at the end of the chapter. A preliminary interpretation of these is given in Chapter 2, in terms of abelian group actions on a Grassmann manifold. This anticipates the Sato–Segal–Wilson approach to KP $\tau$-functions, whose detailed development begins in Chapters 3 and 4 and continues throughout the remainder of the book.

## 1.1 The pendulum and the KdV equation: elliptic function solutions

### *1.1.1 The pendulum*

Consider the motion of a simple pendulum, consisting of a point mass $m$ suspended on a massless rigid rod of length $L$, subject to the force of gravity (Fig. 1.1). The Lagrangian of the system, expressed in terms of the angle $\phi$ from the vertical, is the difference between kinetic and potential energies:

$$L(\phi, \dot{\phi}) = \frac{1}{2} m L^2 \dot{\phi}^2 - 2mgL \sin^2 \frac{\phi}{2}, \tag{1.1.1}$$

where $g$ is the acceleration due to gravity.

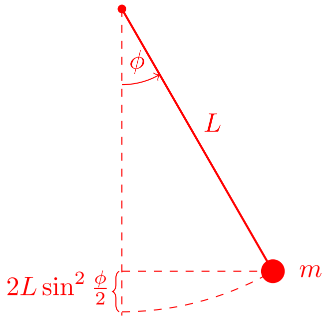


Fig. 1.1. The pendulum

The total energy, which is conserved, is the sum

$$E = \frac{1}{2} m L^2 \dot{\phi}^2 + 2mgL \sin^2 \frac{\phi}{2}. \tag{1.1.2}$$

Introducing the coordinate

$$q =: \sin\frac{\phi}{2}, \tag{1.1.3}$$

we have

$$\dot{q} = \frac{1}{2}\cos\frac{\phi}{2}\dot{\phi} = \frac{1}{2}\sqrt{1-q^2}\,\dot{\phi}, \tag{1.1.4}$$

and the Lagrangian takes the form

$$L(q,\dot{q}) = 2mL^2\frac{\dot{q}^2}{1-q^2} - 2mgLq^2. \tag{1.1.5}$$

The momentum conjugate to $q$ is

$$p := \frac{\partial L}{\partial \dot{q}} = 4mL^2\frac{\dot{q}}{1-q^2}, \tag{1.1.6}$$

and the Legendre transformation gives the Hamiltonian as the sum of kinetic and potential energies:

$$H(q,p) = \frac{1}{8mL^2}(1-q^2)p^2 + 2mgLq^2. \tag{1.1.7}$$

Since the system is autonomous, the total energy is constant

$$\frac{1}{8mL^2}(1-q^2)p^2 + 2mgLq^2 \ = E \tag{1.1.8}$$

and a first integral of the equations of motion is given by its level curves. Substituting (1.1.6), this can be integrated directly, giving $q(t)$ implicitly in terms of an elliptic integral.

It is worthwhile, however, to also consider the problem using the Hamilton–Jacobi method. For this, we define a new momentum variable $P$, which is a constant of motion, by

$$2mgLP^2 := H(q,p), \tag{1.1.9}$$

and seek a generating function $S(q,P)$, Hamilton's principal function, for the transformation from $(q,p)$ to new canonical coordinates $(Q,P)$ in which the equations of motion are trivial. The transformation is defined by

$$p = \frac{\partial S}{\partial q}, \quad Q = \frac{\partial S}{\partial P}, \tag{1.1.10}$$

and the Hamilton–Jacobi equation is

$$H\left(q, \frac{\partial S}{\partial q}\right) = E \tag{1.1.11}$$

or, more explicitly,

$$\left(\frac{\partial S}{\partial q}\right)^2 = 16m^2gL^3\frac{P^2-q^2}{1-q^2}. \tag{1.1.12}$$

The solution is given by an elliptic integral of the second kind:

$$S(q,P) = 4m\sqrt{gL^3}\int_{q_0}^{q}\sqrt{\frac{P^2-x^2}{1-x^2}}dx, \tag{1.1.13}$$

where the constant of integration is absorbed into the choice of initial point $q_0$. The coordinate canonically conjugate to $P$ is thus given by an elliptic integral of the first kind

$$Q = 4m\sqrt{gL^3}P\int_{q_0}^{q}\frac{dx}{\sqrt{(1-x^2)(P^2-x^2)}}, \tag{1.1.14}$$

defined on the curve

$$z^2 = (1-x^2)(P^2-x^2). \tag{1.1.15}$$

In the canonical coordinates $(Q,P)$, the equations of motion have the trivial form

$$\frac{dP}{dt} = -\frac{\partial H}{\partial Q} = 0, \tag{1.1.16}$$

$$\frac{dQ}{dt} = \frac{\partial H}{\partial P} = 4mgLP, \tag{1.1.17}$$

which, when integrated, give a linear flow in time

$$Q(t) = Q_0 + 4mgLPt, \quad P(t) = P_0. \tag{1.1.18}$$

Viewing Hamilton's characteristic function $S(q,P)$ as a function of time, evaluated on the energy level sets, we have

$$S(q(t),P) = 4mL\sqrt{gL}\int_{q(0)}^{q(t)}\sqrt{\frac{P^2-x^2}{1-x^2}}dx, \tag{1.1.19}$$

Changing the integration variable in eq. (1.1.13) from $x$ to $y := x^2$ gives

$$\int_{v_0}^{v(t)}\frac{dy}{\sqrt{y(y-1)(y-e)}} = 2\sqrt{\frac{g}{L}}t, \tag{1.1.20}$$

where

$$v(t) := q^2(t), \quad v_0 := v(0), \quad e := P^2 \tag{1.1.21}$$

and

$$S(q(t),P) = 2mL\sqrt{gL}\int_{v_0}^{v(t)}\sqrt{\frac{e-y}{y(1-y)}}dy. \tag{1.1.22}$$

Introducing the rescaled, translated function

$$u(t) = v\left(\sqrt{\frac{L}{g}}t\right) - \frac{e+1}{3}, \tag{1.1.23}$$

the inverse of the elliptic integral in (1.1.20) becomes a first order differential equation in standard Weierstrass form:

$$(u')^2 = 4u^3 - g_2 u - g_3, \tag{1.1.24}$$

with coefficients

$$g_2 = \frac{4}{3}(e^2 - e + 1), \qquad g_3 = \frac{4}{27}(e+1)(e-2)(2e-1). \tag{1.1.25}$$

The general solution to (1.1.24) is given by the *Weierstrass* $\wp$*-function*

$$u(t) = \wp(t - t_0) \tag{1.1.26}$$

for these parameter values, and any initial value constant $t_0 \in \mathbf{C}$.

In general, $\wp$ is defined by

$$\wp(z) := \frac{1}{z^2} + \sum_{w \in \mathbf{L}\backslash\{0\}} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2}\right], \tag{1.1.27}$$

where the sum is over the integer lattice $\mathbf{L}$ in the complex plane

$$\mathbf{L} = \{2m\omega_1 + 2n\omega_2 \ : \ m, n \in \mathbf{Z}\} \tag{1.1.28}$$

generated by any non-collinear pair of elliptic periods $(2\omega_1, 2\omega_2 \in \mathbf{C}^+)$. This satisfies the Weierstrass equation [280]

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{1.1.29}$$

for modular constants $(g_2, g_3)$ determined from the lattice periods by the Eisenstein series

$$g_2 = 60 \sum_{w \in \mathbf{L}\backslash\{0\}} \frac{1}{w^4}, \qquad g_3 = 140 \sum_{w \in \mathbf{L}\backslash\{0\}} \frac{1}{w^6}. \tag{1.1.30}$$

It can also be expressed as a second logarithmic derivative:

$$\wp(z) = -\frac{d^2}{dz^2} \ln \sigma(z) \tag{1.1.31}$$

in terms of the Weierstrass $\sigma$-function

$$\sigma(z) := z \prod_{w \in \mathbf{L}\backslash\{0\}} \left(1 - \frac{z}{w}\right) \exp\left(\frac{z}{w} + \frac{z^2}{2w^2}\right). \tag{1.1.32}$$

Taking the first derivative of (1.1.29) gives

$$\wp''(t) = 6\wp^2(t) - \frac{g_2}{2}. \tag{1.1.33}$$

Substituting (1.1.31) in (1.1.33) gives the equation of motion in bilinear form in terms of $\sigma$

$$\sigma\sigma'''' - 4\sigma'\sigma''' + 3(\sigma'')^2 - \frac{g_2}{2}\sigma^2 = 0, \tag{1.1.34}$$

where $' := \frac{d}{dt}$. Equivalently, (1.1.34) may be expressed in a more symmetrical way [74] as

$$(\Delta^4 - g_2)\left(\sigma(t-t_0)\sigma(t'-t_0)\right)|_{t=t'} = 0, \tag{1.1.35}$$

where

$$\Delta := \frac{d}{dt} - \frac{d}{dt'}. \tag{1.1.36}$$

Differentiating Hamilton's characteristic function (1.1.19) with respect to $t$ gives

$$\frac{\partial S(q(t), P)}{\partial t} = 4mgL^2 \wp(t - t_0) + \mathcal{E} \tag{1.1.37}$$

where

$$\mathcal{E} := \frac{4}{3}(mgL^2 - E). \tag{1.1.38}$$

So, within an integration constant, we have

$$S(q(t), P) = -\frac{\partial(\ln \sigma)}{\partial t} + \mathcal{E}t. \tag{1.1.39}$$

**Remark 1.1.1.** *The logarithmic derivative formula (1.1.31) expressing the general solution $u(t)$ in terms of the Weierstrass $\sigma$-function will reappear in subsequent examples, as will the bilinear form (1.1.35) of the equation it satisfies. This is the first example of a $\tau$-function generating the solution of an integrable nonlinear equation. It is seen here as closely related to Hamilton's characteristic function $S(q(t), P)$; i.e., the complete solution of the Hamilton–Jacobi equation evaluated on the level sets of the conserved quantities.*

### *1.1.2 Travelling wave solutions of the KdV equation*

The Weierstrass $\wp$-function also appears in another context relating to integrable systems: travelling wave solutions of the nonlinear partial differential equation

$$4u_t = 6uu_x + u_{xxx}, \tag{1.1.40}$$

known as the *Korteweg–de Vries* (KdV) equation, which describes nondissipative shallow water waves in a narrow channel*. Choosing $u(x, t)$ to have the form of a travelling wave

$$u(x, t) = U(x + ct), \tag{1.1.41}$$

* The renewed study of the KdV equation, started in the mid 1960's, led to the discovery of solitons, the inverse scattering method [95–98] and the subsequent flood of interest in completely integrable systems with infinite degrees of freedom.

where $U$ is a function of a single variable $z := x + ct$, and $c$ is the velocity, the KdV equation reduces to the ODE

$$4cU' = 6UU' + U'''. \tag{1.1.42}$$

Integration and multiplication by $U'$ gives

$$4cUU' = 3U^2U' + U''U' + \alpha U', \tag{1.1.43}$$

where $\alpha$ is an integration constant, which can again be integrated to give the first order equation

$$2cU^2 = U^3 + \frac{1}{2}(U')^2 + \alpha U + \beta, \tag{1.1.44}$$

where $\beta$ is a second constant of integration. This can now be reduced to the Weierstrass standard form by the substitution

$$U(z) = -2\wp(z + z_0) + \frac{2c}{3}, \tag{1.1.45}$$

where $z_0 \in \mathbf{C}$ is an arbitrary constant and the modular forms $(g_2, g_3)$ determining $\wp(z)$ are

$$g_2 = \frac{4c^2}{3} - \frac{\alpha}{2} \quad \text{and} \quad g_3 = -\frac{8c^3}{27} + \frac{c\alpha}{6} + \frac{\beta}{4}. \tag{1.1.46}$$

The formula

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\left[e^{\frac{c}{6}x^2}\sigma(x + ct + z_0)\right] \tag{1.1.47}$$

thus gives the general travelling wave solution to the KdV equation, a simple example of an elliptic function solution to a nonlinear evolution equation.

The function

$$\tau(x,t) := e^{\frac{c}{6}x^2}\sigma(x + ct + z_0) \tag{1.1.48}$$

$$= Ke^{\frac{c}{6}x^2}e^{\frac{\eta_1}{2\omega_1}(x+ct+z_0)^2}\theta\left(\frac{x + ct + z_0}{2\omega_1} + \frac{1}{2} + \frac{\omega_2}{2\omega_1}; \frac{\omega_2}{\omega_1}\right) \tag{1.1.49}$$

where $\theta(z;\tau)$ is the Jacobi $\theta$ function

$$\theta(z;\tau) := \sum_{n\in\mathbf{Z}} e^{\pi i\tau n^2 + 2\pi izn}, \qquad \tau := \frac{\omega_2}{\omega_1}, \tag{1.1.50}$$

$K$ is a nonzero constant and

$$\eta_1 := \frac{\sigma'(\omega_1)}{\sigma(\omega_1)} \tag{1.1.51}$$

is another example of a $\tau$-function that, in this case, generates the elliptic function solution of the KdV equation representing generic travelling waves for this case.

### 1.1.3 Degeneration to the trigonometric/hyperbolic case: the separatrix

In terms of the pendulum, the elliptic integral (1.1.20) degenerates to a trigonometric one at the critical energy

$$E_{crit} = 2mgL, \tag{1.1.52}$$

and therefore the solution, either for the pendulum or the travelling wave of the KdV equation, can be written in terms of elementary trigonometric/hyperbolic functions. The corresponding solution of the pendulum problem is known as the *separatrix*, i.e., the special level curve $E = E_{crit}$ of the energy on the phase space of the pendulum in the coordinates $(\phi, \dot{\phi})$. (See Fig. 1.2, where the separatrix $E = E_{crit}$ is indicated.)

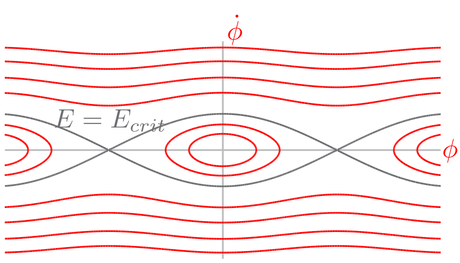


Fig. 1.2. Level curves of the energy $E$ of a pendulum in the $(\phi, \dot{\phi})$ plane

Similarly, if both integration constants $\alpha$ and $\beta$ in (1.1.44) are chosen to be zero, the discriminant for the Weierstrass equation vanishes:

$$\Delta = g_2^3 - 27g_3^2 = 0, \tag{1.1.53}$$

and the general solution to (1.1.44) can be obtained by using elementary hyperbolic functions:

$$U(z) = 2c\,\mathrm{sech}^2(\sqrt{c}(z + z_0)). \tag{1.1.54}$$

Since $U(z)$ can be written as a second logarithmic derivative

$$U(z) = 2\frac{d^2}{dz^2}\ln\cosh(\sqrt{c}(z + z_0)), \tag{1.1.55}$$

the corresponding solution to the KdV equation can be represented as

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\left[\cosh(\sqrt{c}(x + ct + z_0))\right], \tag{1.1.56}$$

which is known as the *one-soliton solution* to the KdV equation, associated to the simple exponential type of $\tau$-function

$$\tau(x,t) = \cosh(\sqrt{c}(x + ct + z_0)) = \frac{1}{2}\left(e^{\sqrt{c}(x+ct+z_0)} + e^{-\sqrt{c}(x+ct+z_0)}\right). \tag{1.1.57}$$

## 1.2 Multisoliton solutions of KdV and KP

For a given positive integer $N$, choose $2N$ complex numbers

$$\{\alpha_k\}_{k=1,\dots,N} \text{ and } \{\gamma_k\}_{k=1,\dots,N} \tag{1.2.1}$$

with all $\alpha_k$'s pairwise distinct and all $\gamma_k$'s nonzero. Define $N$ functions

$$y_k(\mathbf{t}) := e^{\sum_{i=1}^\infty t_i \alpha_k^i} + \gamma_k e^{\sum_{i=1}^\infty t_i (-\alpha_k)^i}, \quad k = 1, \dots, N, \tag{1.2.2}$$

where $\mathbf{t}$ is an infinite sequence of variables

$$\mathbf{t} = (t_1, t_2, \dots), \tag{1.2.3}$$

referred to as the higher KdV flow variables or *times*. Note that

$$y_k^{(l)}(\mathbf{t}) := \frac{\partial^l}{\partial t_1^l} y_k(\mathbf{t}) = (\alpha_k)^l \left[ e^{\sum_{i=1}^\infty t_i \alpha_k^i} + (-1)^l \gamma_k e^{\sum_{i=1}^\infty t_i (-\alpha_k)^i} \right] \tag{1.2.4}$$

$$= 2(\alpha_k)^l \gamma_k^{1/2} e^{\sum_{i=1}^\infty t_{2i} \alpha_k^{2i}} \begin{cases} \cosh\left( \sum_{i=0}^\infty t_{2i+1} \alpha_k^{2i+1} - \frac{1}{2} \log \gamma_k \right) & l \text{ even} \\ \sinh\left( \sum_{i=0}^\infty t_{2i+1} \alpha_k^{2i+1} - \frac{1}{2} \log \gamma_k \right) & l \text{ odd.} \end{cases} \tag{1.2.5}$$

Now define the $\tau$-function $\tau^{(N)}_{\alpha_1,\dots,\alpha_N,\gamma_1,\dots,\gamma_N}(\mathbf{t})$ as the Wronskian determinant

$$\tau^{(N)}_{\alpha_1,\dots,\alpha_N,\gamma_1,\dots,\gamma_N}(\mathbf{t}) := \begin{vmatrix} y_1(\mathbf{t}) & y_2(\mathbf{t}) & \cdots & y_N(\mathbf{t}) \\ y_1'(\mathbf{t}) & y_2'(\mathbf{t}) & \cdots & y_N'(\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(N-1)}(\mathbf{t}) & y_2^{(N-1)}(\mathbf{t}) & \cdots & y_N^{(N-1)}(\mathbf{t}) \end{vmatrix} \tag{1.2.6}$$

$$= e^{\sum_{i=1}^\infty \sum_{k=1}^N \alpha_k^{2i} t_{2i}} \begin{vmatrix} y_1(\mathbf{t}_0) & y_2(\mathbf{t}_0) & \cdots & y_N(\mathbf{t}_0) \\ y_1'(\mathbf{t}_0) & y_2'(\mathbf{t}_0) & \cdots & y_N'(\mathbf{t}_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(N-1)}(\mathbf{t}_0) & y_2^{(N-1)}(\mathbf{t}_0) & \cdots & y_N^{(N-1)}(\mathbf{t}_0) \end{vmatrix},$$

where $\mathbf{t}_0 := (t_1, 0, t_3, 0, \dots)$, and the derivatives $\{{y_i}' \dots {y_i}^{(N-1)}\}$ are taken with respect to $x = t_1$. The function $\tau^{(N)}_{\alpha_1,\dots,\alpha_N,\gamma_1,\dots,\gamma_N}(\mathbf{t})$ has the remarkable property that twice its second logarithmic derivative, evaluated at the parameter values $(t_1 = x, t_2 = 0, t_3 = t,\ t_i = 0,\ i > 3)$

$$u(x,t) := 2\frac{\partial^2}{\partial x^2} \log \tau^{(N)}_{\alpha_1,\dots,\alpha_N,\gamma_1,\dots,\gamma_N}(x, 0, t, 0, 0 \dots) \tag{1.2.7}$$

satisfies the KdV equation (1.1.40). Solutions of this form are called *standard $N$-soliton solutions* to the KdV equation.

More generally, if we choose $3N$ complex constants

$$\{\alpha_k, \beta_k, \gamma_k\}_{k=1,\dots,N} \tag{1.2.8}$$

with $\alpha_k, \beta_k$'s all distinct, $\gamma_k \neq 0$, and define the functions

$$y_k(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i \alpha_k^i} + \gamma_k e^{\sum_{i=1}^{\infty} t_i \beta_k^i}, \quad k = 1, \dots, N, \tag{1.2.9}$$

we arrive at the more general Wronskian determinant

$$\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(\mathbf{t}) := \begin{vmatrix} y_1(\mathbf{t}) & y_2(\mathbf{t}) & \cdots & y_N(\mathbf{t}) \\ y_1'(\mathbf{t}) & y_2'(\mathbf{t}) & \cdots & y_N'(\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(N-1)}(\mathbf{t}) & y_2^{(N-1)}(\mathbf{t}) & \cdots & y_N^{(N-1)}(\mathbf{t}) \end{vmatrix}. \tag{1.2.10}$$

The function

$$u(x, y, t) := 2\frac{\partial^2}{\partial x^2} \log\left(\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(x, y, t, t_4, \dots)\right) \tag{1.2.11}$$

can then be shown to satisfy the $2+1$ dimensional nonlinear partial differential equation

$$3u_{yy} = (4u_t - 6uu_x - u_{xxx})_x, \tag{1.2.12}$$

known as the *Kadomtsev–Petviashvili* (KP) *equation* (which plays a prominent rôle in plasma physics and in the study of shallow water ocean waves), together with an infinite set of further nonlinear autonomous PDEs, each involving partial derivatives of finite order with respect to a finite number of the KP flow parameters $\mathbf{t} = (t_1, t_2, \dots)$. These are collectively known as the *KP hierarchy.* They may all be deduced from a single family of bilinear relations known as the Hirota bilinear equations, (see Section 1.10 below), satisfied by the $\tau$-function $\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(\mathbf{t})$, and by all solutions of the KP hierarchy. Solutions of the form (1.2.10) are referred to as *standard $N$-soliton solutions* of the KP hierarchy in Wronskian form.

If $\beta_j = -\alpha_j$ for all $j$, the standard KP-solitons are independent of $y = t_2$ and all further even flow parameters $\{t_{2i}\}$ and reduce to KdV $N$-solitons, since

$$e^{x\alpha_k + y\alpha_k^2 + t\alpha_k^3} + \gamma_k e^{-x\alpha_k + y\alpha_k^2 - t\alpha_k^3} = e^{y\alpha_k^2}\left[e^{x\alpha_k + t\alpha_k^3} + \gamma_k e^{-x\alpha_k - t\alpha_k^3}\right], \tag{1.2.13}$$

and the second logarithmic derivative in $x$ eliminates the $y$-dependence.

The Wronskian formula (1.2.10) for the $\tau$-function $\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(\mathbf{t})$ can also be rewritten in a more general determinantal form [102, 104] (detailed in Section 6.1) as

$$\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(\mathbf{t}) = \det(A e^{\sum_{i=1}^{\infty} t_i B^i} C^T), \tag{1.2.14}$$

where, for (1.2.10), $A$ is the $N \times 2N$ double Vandermonde-type matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_N & \beta_1 & \beta_2 & \cdots & \beta_N \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{N-1} & \alpha_2^{N-1} & \cdots & \alpha_N^{N-1} & \beta_1^{N-1} & \beta_2^{N-1} & \cdots & \beta_N^{N-1} \end{bmatrix}, \tag{1.2.15}$$