# A First Course in General Relativity

## Third Edition

Clarity, readability, and rigor combine in the third edition of this widely used textbook to provide the first step into general relativity for advanced undergraduate students with a minimal background in mathematics. Topics within relativity that fascinate astrophysics researchers and students alike are covered with Schutz's characteristic ease and authority, from black holes to relativistic objects, from pulsars to the study of the Universe as a whole. This third edition contains discoveries by astronomers that require general relativity for their explanation; two chapters on gravitational waves, including direct detections of gravitational waves and their observations' impact on cosmological measurements; new information on black holes and neutron stars; and greater insight into the expansion of the Universe. Over 300 exercises, many new to this edition, give students the confidence to work with general relativity and the necessary mathematics, while the informal writing style and worked examples make the subject matter easily accessible.

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# Preface to the third edition

This third edition follows the second after 12 years, half of the interval between the first and second editions. The need for a new edition is a happy one: the direct detections of gravitational waves, beginning in 2015, have already made much of the second edition obsolete. These detections have surprised astronomers by revealing how unexpectedly plentiful and massive are the black holes that form when stars die; they have demonstrated that neutron stars are in fact our direct ancestors because their collisions created most of the heaviest elements in our bodies; and they have given us a new and potentially powerful way of measuring the expansion rate and age of the Universe.

This new edition incorporates much of what we have learned since 2015, but more importantly it aims to provide a foundation that should enable the reader to understand future observations and the discoveries they bring. In the preface to the previous edition I noted that general relativity had generated “a host of applications, some of which were not even imagined in 1985 when the first edition appeared.” Now there has been another big step: with gravitational wave detections, general relativity has added a radical new way of observing the Universe, a way of learning about things that we could not have investigated before.

Gravitational wave astronomy is now the fastest expanding branch of observational astronomy: the volume of space accessible to the current detectors has doubled on average every 1.5 years since 2015, as their sensitivity has improved. This rate looks set to continue for some years as new detectors in Japan and India join and steadily improve to full sensitivity. The space-based LISA detector is under construction; it will be able to detect events at distances far greater than the range of any optical telescope. And new detectors are being proposed that can register every black-hole and neutron-star merger back to times equal to the age of our Universe. At the same time, gravitational wave detection is being integrated into other observing fields, as large survey telescopes like the Vera Rubin Observatory provide the capability of searching for explosions that may accompany gravitational wave events, as the Athena X-ray satellite observatory mission is being developed to do coincident observing with LISA, as radio astronomers use pulsars as gravitational wave detectors, and as microwave astronomers search for the signature of gravitational waves in the very early Universe.

The changes in this textbook for the third edition have been designed in recognition of this sea-change, anticipating what students want to know and what future researchers need to be prepared for. The final four chapters have been expanded to five, and all have been rewritten. They embrace the new role of general relativity in astronomy in several ways.

- There are now two chapters on gravitational waves. They treat the theory, survey the detectors current and planned, provide details of the most significant detections so far, and explain the basics of how to extract and interpret information from observations now and in the future.
- The chapters on stars and black holes now incorporate what we have learned from the gravitational wave observations to date. In addition, there is much new material on the astronomical context of these observations, such as how stellar evolution produces neutron stars and black holes. The aim is to help readers to fit gravitational wave observations of relativistic objects into the bigger picture.
- The chapter on cosmology – the study of the Universe as a whole – has been completely reorganized and greatly expanded. Cosmology is another rapidly moving and fundamental part of astronomy today and, on top of that, gravitational wave observations have provided it with a new way of measuring distances over vast reaches of space. The chapter now starts with what we observe, because cosmology is now a high-precision observational science. It then goes on to apply Einstein's equations to explain the observations and ends with cosmology's frontiers in fundamental physics.

The mathematical foundations of general relativity, treated in the first eight chapters, remain as solid as before: general relativity is now a rigorously tested theory, with some experiments and observations unable to find deviations from its predictions even at levels of one part in $10^{15}$ or better. So the revisions for this edition of the first eight chapters have been minor: modernizing some of the treatment, correcting typos, simplifying some explanations.

One aspect preserved from earlier editions is that this textbook remains one of the few at this level that offers readers accessible insights into a selection of key topics that are normally avoided or simply quoted without derivation. These include a physical derivation of the energy flux carried by gravitational waves (an elaboration of Feynman's intuitive 1957 bead-on-a-wire argument for the physical reality of gravitational waves); an order-of-magnitude derivation of the Hawking radiation from the starting point of quantum fluctuations outside the horizon of a black hole; a full discussion of equatorial orbits in the Kerr metric, using a factorizable potential; and a treatment of the elegant Forward–Berman description of inspiralling binaries that shows that they are standard sirens because general relativity is a scale-free theory.

This book remains, of course, a beginner's introduction to the science of general relativity. All the new developments can be studied in much greater depth in many new books, and so the bibliography at the end of each chapter has been updated to point readers to references that can take them further in whatever direction they may find interesting.

I have benefitted from helpful conversations and input from many colleagues, both within the gravitational wave community and in the wider astronomy community. I thank them all, too many to name. But I also wish to thank those readers of previous editions who have very kindly drawn my attention to typos and other issues, particularly Tom Bartholet, Alon Brook-Ray, Sami Gara, Aleksandr Pargamotnikas, and Leo Schirber. Any remaining errors are, of course, my own responsibility. I thank also my editors at Cambridge University Press, Simon Capelin, Vince Higgs, and Ilaria Tassistro, for their patience and encouragement this third time around.

# Preface to the second edition

In the twenty-three years between the first edition of this textbook and the present revision, the field of general relativity has blossomed and matured. Upon its solid mathematical foundations have grown a host of applications, some of which were not even imagined in 1985 when the first edition appeared. The study of general relativity has therefore moved from the periphery to the core of the education of a professional theoretical physicist, and more and more undergraduates expect to learn at least the basics of general relativity before they graduate.

My readers have been patient. Students have continued to use the first edition of this book to learn about the mathematical foundations of general relativity, even though it has become seriously out of date on applications like the astrophysics of black holes, the detection of gravitational waves, and the exploration of the Universe. This extensively revised second edition will, I hope, finally bring the book back into balance and give readers a consistent and unified introduction to modern research in classical gravitation.

The first eight chapters have seen little change. Recent references for further reading have been included, and a few sections have been expanded, but in general the geometrical approach to the mathematical foundations of the theory seems to have stood the test of time. By contrast, the final four chapters, which deal with general relativity in the astrophysical arena, have been updated, expanded, and in some cases completely re-written.

In Chapter 9, on gravitational radiation, there is now an extensive discussion of detection with interferometers like LIGO and the planned space-based detector LISA. I have also included a discussion of likely gravitational wave sources, and what we can expect to learn from detections. This is a field that is rapidly changing, and the first-ever direct detection could come at any time. Chapter 9 is intended to provide a durable framework for understanding the implications of these detections.

In Chapter 10, the discussion of the structure of spherical stars remains robust, but I have inserted material on real neutron stars, which we see as pulsars and which are potential sources of detectable gravitational waves.

Chapter 11, on black holes, has also gained extensive material about the astrophysical evidence for black holes, both for stellar-mass black holes and for the supermassive black holes that astronomers have astonishingly discovered in the centers of most galaxies. The discussion of the Hawking radiation has also been slightly amended.

Finally, Chapter 12 on cosmology is completely rewritten. In the first edition I essentially ignored the cosmological constant. In this I followed the prejudice of the time, which assumed that the expansion of the Universe was slowing down, even though it had not yet been accurately enough measured. We now know, from a variety of mutually consistent observations, that the expansion is accelerating. This is probably the biggest challenge to theoretical physics today, having an impact as great on fundamental theories of particle physics as on cosmological questions. I have organized Chapter 12 around this fact, developing mathematical models of an expanding Universe that include the cosmological constant, then discussing in detail how astronomers measure the kinematics of the Universe, and finally exploring the way that the physical constituents of the Universe evolved after the Big Bang. The roles of inflation, of dark matter, and of dark energy all affect the structure of the Universe today, and even our very existence. In this chapter it is possible only to give a brief taste of what astronomers have learned about these issues, but I hope it is enough to encourage readers to go on to learn more.

I have included more exercises in various chapters, where it was appropriate, but I have removed the exercise solutions from the book. They are available now on the website for the book.

The subject of this book remains classical general relativity: apart from a brief discussion of the Hawking radiation, there is no reference to quantization effects. While quantum gravity is one of the most active areas of research in theoretical physics today, there is still no clear direction to point a student who wants to learn how to quantize gravity. Perhaps by the third edition it will be possible to include a chapter on how gravity is quantized!

I want to thank many people who have helped me with this second edition. Several have generously supplied me with lists of misprints and errors in the first edition; I especially want to mention Frode Appel, Robert D'Alessandro, J. A. D. Ewart, Steve Fulling, Toshi Futamase, Gerald Quinlan, and B. Sathyaprakash. Any remaining errors are, of course, my own responsibility. I thank also my editors at Cambridge University Press, Rufus Neal, Simon Capelin, and Lindsay Barnes, for their patience and encouragement. And of course I am deeply indebted to my wife Sîan for her generous patience during all the hours, days, and weeks I spent working on this revision.

# Preface to the first edition

This book has evolved from lecture notes for a full-year undergraduate course in general relativity which I taught from 1975 to 1980, an experience which firmly convinced me that general relativity is not significantly more difficult for undergraduates to learn than the standard undergraduate-level treatments of electromagnetism and quantum mechanics. The explosion of research interest in general relativity in the past 20 years, largely stimulated by astronomy, has not only led to a deeper and more complete understanding of the theory; it has also taught us simpler, more physical ways of understanding it. Relativity is now in the mainstream of physics and astronomy, so that no theoretical physicist can be regarded as broadly educated without some training in the subject. The formidable reputation relativity acquired in its early years (Interviewer: 'Professor Eddington, is it true that only three people in the world understand Einstein's theory?' Eddington: 'Who is the third?') is today perhaps the chief obstacle that prevents it being more widely taught to theoretical physicists. The aim of this textbook is to present general relativity at a level appropriate for undergraduates, so that the student will understand the basic physical concepts and their experimental implications, will be able to solve elementary problems, and will be well prepared for the more advanced texts on the subject.

In pursuing this aim, I have tried to satisfy two competing criteria: first, to assume a minimum of prerequisites; and second, to avoid watering down the subject matter. Unlike most introductory texts, this one does not assume that the student has already studied electromagnetism in its manifestly relativistic formulation, the theory of electromagnetic waves, or fluid dynamics. The necessary fluid dynamics is developed in the relevant chapters. The main consequence of not assuming a familiarity with electromagnetic waves is that gravitational waves have to be introduced slowly: the wave equation is studied from scratch. A full list of prerequisites appears below.

The second guiding principle, that of not watering down the treatment, is very subjective and rather more difficult to describe. I have tried to introduce differential geometry fully, not being content to rely only on analogies with curved surfaces, but I have left out subjects that are not essential to general relativity at this level, such as nonmetric manifold theory, Lie derivatives, and fiber bundles.[1] I have introduced the full nonlinear field equations, not just those of linearized theory, but I solve them only in the plane and spherical cases, quoting and examining, in addition, the Kerr solution. I study gravitational waves mainly in the linear approximation, but go slightly beyond it to derive the energy in the waves and the reaction effects in the wave emitter. I have tried in each topic to supply enough

[1] The treatment here is therefore different in spirit from that in my book *Geometrical Methods of Mathematical Physics* (Cambridge University Press 1980b), which may be used to supplement this one.

foundation for the student to be able to go to more advanced treatments without having to start over again at the beginning.

The first part of the book, up to Chapter 8, introduces the theory in a sequence which is typical of many treatments: a review of special relativity, development of tensor analysis and continuum physics in special relativity, study of tensor calculus in curvilinear coordinates in Euclidean and Minkowski spaces, geometry of curved manifolds, physics in a curved spacetime, and finally the field equations. The remaining four chapters study a few topics which I have chosen because of their importance in modern astrophysics. The chapter on gravitational radiation is more detailed than usual at this level because the observation of gravitational waves may be one of the most significant developments in astronomy in the next decade. The chapter on spherical stars includes, besides the usual material, a useful family of exact compressible solutions due to Buchdahl. A long chapter on black holes studies in some detail the physical nature of the horizon, going as far as the Kruskal coordinates, then exploring the rotating (Kerr) black hole, and concluding with a simple discussion of the Hawking effect, the quantum mechanical emission of radiation by black holes. The concluding chapter on cosmology derives the homogeneous and isotropic metrics and briefly studies the physics of cosmological observation and evolution. There is an appendix summarizing the linear algebra needed in the text, and another appendix containing hints and solutions for selected exercises. One subject I have decided not to give as much prominence to as other texts traditionally have is experimental tests of general relativity and of alternative theories of gravity. Points of contact with experiment are treated as they arise, but systematic discussions of tests now require whole books (Will 1981, 2nd edn 2018). Physicists today have far more confidence in the validity of general relativity than they had a decade or two ago, and I believe that an extensive discussion of alternative theories is therefore almost as out of place in a modern elementary text on gravity as it would be in one on electromagnetism.

The student is assumed already to have studied: special relativity, including the Lorentz transformation and relativistic mechanics; Euclidean vector calculus; ordinary and simple partial differential equations; thermodynamics and hydrostatics; Newtonian gravity (simple stellar structure would be useful but not essential); and enough elementary quantum mechanics to know what a photon is.

The notation and conventions are essentially the same as in Misner, Thorne, & Wheeler, *Gravitation* (W. H. Freeman 1973), which may be regarded as one possible follow-on text after this one. The physical point of view and development of the subject are also inevitably influenced by that book, partly because Thorne was my teacher and partly because *Gravitation* has become such an influential text. But because I have tried to make the subject accessible to a much wider audience, the style and pedagogical method of the present book are very different.

Regarding the use of the book, it is designed to be studied sequentially as a whole, in a one-year course, but it can be shortened to accommodate a half-year course. Half-year courses probably should aim at restricted goals. For example, it would be reasonable to aim to teach gravitational waves and black holes in half a year to students who have already studied electromagnetic waves, by carefully skipping some of Chapters 1–3 and most of Chapters 4, 7, and 10. Students with preparation in special relativity and fluid dynamics could learn stellar structure and cosmology in half a year, provided they could go quickly through the first four chapters and then skip Chapters 9 and 11. A graduate-level course can, of course, go much more quickly, and it should be possible to cover the whole text in half a year.

Each chapter is followed by a set of exercises, which range from trivial ones (filling in missing steps in the body of the text, manipulating newly introduced mathematics) to advanced problems that considerably extend the discussion in the text. Some problems require programmable calculators or computers. I cannot overstress the importance of doing a selection of problems. The easy and medium-hard ones in the early chapters give essential practice, without which the later chapters will be much less comprehensible. The medium-hard and hard problems of the later chapters are a test of the student's understanding. It is all too common in relativity for students to find the conceptual framework so interesting that they relegate problem solving to second place. Such a separation is false and dangerous: a student who can't solve problems of reasonable difficulty doesn't really understand the concepts of the theory either. There are generally more problems than one would expect a student to solve; several chapters have more than 30. The teacher will have to select them judiciously. Another rich source of problems is the *Problem Book in Relativity and Gravitation*, Lightman *et al.* (Princeton University Press 1975).

I am indebted to many people for their help, direct and indirect, with this book. I would like especially to thank my undergraduates at University College, Cardiff, whose enthusiasm for the subject and whose patience with the inadequacies of the early lecture notes encouraged me to turn them into a book. And I am certainly grateful to Suzanne Ball, Jane Owen, Margaret Vallender, Pranoat Priesmeyer and Shirley Kemp for their patient typing and retyping of the successive drafts.