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Excerpt
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Introduction

One of the greatest discoveries in mathematics is the Fredholm index. This measures the size of the solution space for a linear system. The beauty of this index is that it is invariant under small perturbations of the linear system.

The topology and geometry of a smooth closed manifold M is governed by certain natural elliptic differential operators. These operators have Fredholm indices that are computed by the famous Atiyah–Singer index formula. The work underlying this formula was one of the foremost mathematical achievements of the last century, and has important applications in geometry, topology, and mathematical physics.

A central question in mathematics is to extend the Atiyah–Singer index theory to non-compact manifolds. In the non-compact case, the classic Fredholm index is not well defined since the solution spaces of natural elliptic differential operators can be infinite-dimensional. A vast generalisation of the Fredholm index, called the higher index, can be defined for differential operators within the framework of Alain Connes' noncommutative geometry. A key idea in the definition of higher index is to develop a notion of dimension for possibly infinite-dimensional spaces using operator algebras. This dimension theory has its root in John von Neumann's theory of continuous geometry and is formalised using K-theory of operator algebras. Crucial features are that the higher index is invariant under small perturbations of the differential operator, and that it is an obstruction to invertibility of the operator.

Higher index theory has been developed in the work of many mathematicians over the last 40 years. It has found fundamental applications to geometry and topology, such as to the Novikov conjecture on topological rigidity and the Gromov–Lawson conjecture on scalar curvature.

The purpose of the book is to give a friendly exposition of this exciting subject!

Structure

This book is split into four parts.

Part ONE summarises background on C^*-algebras and K-theory, often only including full proofs for non-standard material. The reader should not expect to have to understand all of this before approaching the rest of the book. Part ONE ends with a section motivating some of the techniques we will study, based on the problem of existence of positive scalar curvature metrics. We also give a more detailed summary of the book's contents here.

Part TWO discusses Roe algebras, localisation algebras and the assembly maps connecting them. Roe algebras and localisation algebras are C^*-algebras associated to the large- and small-scale structures of a space, and assembly is a map between them. Assembly is closely related to taking higher indices. This section finishes with a description of the Baum–Connes conjectures, which posit that a certain universal assembly map is an isomorphism.

Part THREE moves into the theory of differential operators on manifolds, which is where the main applications of the theory developed in Part TWO lie; in the earlier parts, we do not really discuss manifolds at all. We discuss how elliptic operators naturally give rise to K-theory classes, and the flavour thus becomes more explicitly index-theoretic. We also discuss how Poincaré duality in K-theory relates to differential operators, and summarise some of the most important applications to geometry and topology.

Part FOUR looks at the (Baum–Connes) assembly maps in more detail. We give an elementary approach to some results in the case of almost constant bundles. We spend some time giving a new and relatively elementary proof of the coarse Baum–Connes conjecture for spaces that admit a coarse embedding into Hilbert space, a particularly important theorem for applications. We also discuss some counterexamples.

Finally, the book closes with several appendices summarising an ad hoc collection of material from general topology and coarse geometry, from representation theory, from the theory of unbounded operators, and about graded C^*-algebras and Hilbert spaces.

Intended Audience and Prerequisites

The prerequisites are something like a first course in C^*-algebra K-theory, some of which is summarised in Part ONE. For Part THREE, it will also help to have some background in manifold topology and geometry, although this is generally kept to a minimum.

The intended audience consists of either operator algebraists who are interested in applications of their field to topology and geometry, or topologists and geometers who want to use tools from operator algebras and index theory.

We have done our best to keep the exposition as concrete and direct as possible.

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