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Introduction

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Contents

1 A Brief History of Space

Space is a central notion in both mathematics and physics and has always been at the heart of their interactions. From Greek geometry to Galileo experiences, mathematics and physics have been rooted in constructions performed in the same ambient physical space. But both mathematics and physics have eventually left the safe experience of this common ground for more abstract notions of space.

The 17th century witnessed the development of projective geometry and the strange, yet effective, idea of points at infinity. This century witnessed also the advent of analytic geometry (with its use of coordinates) with Descartes and of differential calculus with Newton and Leibniz. Both have led to an approach to geometry fundamentally based on the manipulation of algebraic formulas. The capacity to manipulate spaces without relying on a spatial intuition has laid the foundations for one of the most important revolutions in geometry: the conception of spaces of arbitrary dimension.

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During the same time, the successful geometrization of astronomy, optics, and mechanics anchored physics to the paradigm of a differential Euclidean space.

The 18th century essentially develops differential calculus for spaces of high dimensions. Analytic geometry and physics converge into analytical mechanics. This is a revolution that introduces abstract spaces in physics (like the six-dimensional spaces of trajectories) and reduces the actual physical space to be a mere starting point out of which other relevant spaces can be constructed. In mathematics, the invention of complex numbers laid the groundwork for the future algebraic geometry. In turn, the contradictions of logarithm theory and the study of polyhedra and graphs planted the seeds of algebraic topology. Moreover, analytical methods forecast a new notion of space: the infinite-dimensional spaces of functions.

In the 19th century, geometry exploded in diversity. The use of local coordinates in analytical mechanics gave rise to the intrinsic theory of manifolds and the fundamental local–global dialectic. The points at infinity, the points with complex coordinates, and the multiple points of intersection theory are all unified in the framework of algebraic geometry. The geometric study of linear equations led to the notion of vector space. The development of Lie group theory created a completely new branch of geometry centered on the characterization of the symmetries of spaces. The construction of models for non-Euclidean geometry revived the old synthetic/axiomatic geometry, and the development of analysis prepares the notions of metric spaces. In physics, thermodynamics and electromagnetism are successfully developed within the framework of the differential calculus of $\mathbb{R}^n$. The latter entangles space with time in an unusual way, but the geometric paradigm of classical mechanics remains well secured.

The mathematics of the 20th century started with the successful definition of topological spaces. At the heart of the notion of space are now the set of its points, the open subsets, and the continuous-discontinuous dialectic. From function spaces to manifolds and unseparated spaces, topological spaces are powerful enough to unify many kinds of spaces. Lie groups and the paradigm of symmetry are also everywhere, from differential equations to manifolds and linear algebra. Another major revolution was the discovery that the spaces of high dimensions have specific shapes and can be different from each other. This qualitative study of spaces gave birth to algebraic topology and its two branches of homotopy and homology theories. With higher-dimensional spaces and algebraic topology, figures have essentially disappeared from geometry books, and geometry has become the study of spaces that cannot be “seen” anymore.

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In physics, the contradictions raised by the constancy of the speed of light and the spectrum of the black body were the source of a schism on the role of space. Relativity grounded the geometry of physics in the new dynamical object that is spacetime and successfully formalized gravity in purely geometrico-differential terms. Yang–Mills theory extended this program to the other (electromagnetic and nuclear) fundamental interactions. On the other side, the formalism of quantum mechanics required abandoning geometric intuition and, rather, focusing on algebras of operators. Despite the fundamental role played by symmetries and Lie group theory in both theories, the geometric unity of physics was to a certain extent broken.

By the middle of the 20th century, mathematics and physics are much better structured than they were at the beginning. The notions of sets, topological spaces, manifolds (Riemannian or not), algebraic varieties, and vector spaces organize the geography of mathematical spaces. General relativity, classical mechanics, and quantum mechanics divide the physical space in three scales, each with its own geometric formalism. The situation is summarized in Tables 0.1 and 0.2. All things seems to fall into place, and, for our purposes in this book, we shall refer to this situation as the *classical paradigm of space*. The conceptual categories that organize this paradigm on the mathematical side are points, open and closed subsets, coordinates and functions, local/global, measure of distances, continuity/discontinuity, infinitesimal variations, and approximation. On the physical side, the classical paradigm relies on a differentiable spacetime, trajectories and fields, infinitesimal equations, and symmetries and covariance. The intuition of space has been pushed far away from the original intuition of the ambient physical space, but in a clear continuity.

The evolution of the notion of space in mathematics and physics has continued until now. However, the results of these developments are less universally known in the mathematical and physical communities where the common background stays, even nowadays, the classical paradigm. It is the purpose of this book and its companion to illustrate and explain some of these "postclassical" developments.

2 Contemporary Mathematical Spaces

2.1 Algebraic Topology

One of the most important geometric achievements of the postclassical period is the revisitation of *algebraic topology* (homotopy and homology theory) in terms of higher category theory. Homotopy theory evolved from the definition

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Table 0.1. *The "classical" kinds of mathematical spaces*

Basic structures	*Topology*	*Differential geometry*	*Linear spaces*	*Algebraic geometry*
Sets	Topological space	Differential manifold, Lie groups	Vector spaces, function spaces, modules over rings	Algebraic variety, algebraic groups
Preorder, equivalence relations	Metric space	Riemannian manifold	Banach, Fréchet, Hilbert... spaces	

Table 0.2. *The "classical" kinds of physical spaces*

	General relativity *(large scale)*	*Mechanics and thermodynamics* *(medium scale)*	*Quantum mechanics* *(small scale)*
Ambient space and time	Lorentzian 4-manifold	Galilean spacetime $\mathbb{R} \times \mathbb{R}^3$	Galilean or Poincaré Lie group
Phase spaces	Spaces with action of the local Poincaré group	Manifolds with action of the Galilean group	Representations of various Lie groups in Hilbert spaces

of the fundamental group of a space (and its applications to classify covering spaces and to explain the multiple values of analytic continuations) to a general study of continuous maps and spaces up to continuous deformations (homotopies and homotopy equivalences) [25, 43]. The central object ended up to be that of the *homotopy type* of a space, that is, the equivalent classes of this space up to homotopy equivalence. Homological algebra evolved from a computation of numbers and groups to a calculus of resolutions of modules over a ring (or sheaves of such) [16, 63]. The notion of abelian category put some order in this calculus [13, 36], but it is only with triangulated categories that a central object emerged: *chain complexes* up to quasi-isomorphisms [23, 93]. In a separated approach, the axiomatization of homology theories in terms of functors had also led to a new kind of object: *spectra*, of which chain complexes are a particular instance [1, 57, 77]. Any space defines both a

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homotopy type and a spectrum (its *stable homotopy type*), but until the 1970s, the nature of these two objects was somehow elusive.

The development in the 1970s of *homotopical algebra* (i.e., model category theory) provided for the first time a unified framework for both homotopy and homology theories [31, 73]. But even with this unification, the theory was still highly technical and, many times, ad hoc. The concepts that revealed the meaning of these constructions were only found in the 1980s, when higher category theory emerged [11, 28, 38, 72]. The main progress was to understand that homotopy types of spaces were the same thing as ∞-groupoids, that is, a particular kind of higher category in which all morphisms are invertible (see Chapter 5). By viewing homotopy types as ∞-groupoids, it was possible to revisit homotopical algebra from the standpoint provided by the whole conceptual apparatus of higher category theory. This has provided conceptual simplification of many of the homotopical constructions, but this story lies beyond the scope of this book (see [18, 61]). We have limited our study to the utilization of ∞-groupoids in geometry, namely, in topos theory (Chapter 4), in stack theory (Chapter 8), and in the theory of derived schemes (Chapter 9). We have also included a chapter explaining how ∞-groupoids have permitted us to revisit the foundations of mathematics (Chapter 6). Moreover, Chapters 4 and 5 of the companion volume, *New Spaces in Physics*, show how ∞-groupoids are useful in symplectic geometry and physics.

2.2 Algebraic Geometry

The field of geometry that has undergone the deepest postclassical transformation is *algebraic geometry*. From the 1950s to the 1980s, Grothendieck's school brought many definitions and improvements for the objects of algebraic geometry. The definition of Zariski spectra and schemes as ringed spaces permitted for the first time the unification of all the notions of algebraic varieties. Moreover, the notion of affine scheme provided a perfect duality between some geometric objects and arbitrary commutative rings of coordinates [39]. An important difference that schemes have with manifolds is the fact that they can accommodate singular points. This singular structure is encoded algebraically by the existence of nilpotent elements in the ring of local coordinates, a feature that is possible only if arbitrary rings are considered. Nilpotent elements provide an efficient infinitesimal calculus, which is one of the nicest achievements of algebraic geometry (see [21] and volume IV of [39]).[1]

[1] This calculus is also at the core of *synthetic differential geometry*; see Chapter 2.

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The theory of ringed spaces was efficient to define general schemes by pasting of affine schemes. However, motivated by the study of algebraic groups and the construction of moduli spaces, schemes were almost immediately redefined as functors, making the previous construction somehow superfluous (see Chapter 7 on the functor of points and [24, 37]). Later on, the definition of étale spectra of rings (which was needed to define cohomology theories with étale descent) came back to a definition in terms of ringed spaces with the difference that the base space was now a *topos* (see Chapter 4 and [5, 23, 64]). The functorial point of view continued to be used simultaneously.[2]

The definitive approach to constructing *moduli spaces* (e.g., the spaces of curves or bundles on a given space) was eventually found with *stacks*, which are a variation on the notion of sheaf (see Chapter 8 and [3, 22, 33, 38, 55, 81]). Essentially, stacks provide a notion of space where the set of points is enhanced into a groupoid of points. This feature makes them perfectly suited to classifying objects (such as curves or bundles) together with their symmetries. From a geometric point of view, stack theory is a formalism intended to deal properly with the possible singularities created by taking a quotient (see Chapter 9).

The most recent development has been *derived algebraic geometry*. This formalism enhances the theory of stacks in order also to tame the singularities created by nontransverse intersections (see Chapter 9 and [62, 86, 87]). At the end of the story, derived algebraic geometry provides by far the most sophisticated notion of space ever invented.[3] Derived stacks have become a powerful archetype for a new paradigm of geometric spaces (see Chapter 9 and [48, 58, 70, 84, 88]). However, so many turns in only 60 years have been hard to follow, and the community of algebraic geometers is largely spread out between different technologies and viewpoints on its objects.

An important field related to algebraic geometry is *complex geometry*. In comparison with their differential analogs, complex manifolds have the problem that they admit too few globally defined holomorphic functions. This has deeply grounded the field in sheaf theory and cohomological methods and kept it close to algebraic geometry, where the same methods were used for similar reasons [34, 44, 79, 94]. Nonetheless, complex manifolds have

[2] For example, the notion of a connection on a singular scheme X was successfully defined by means of the *de Rham shape* of X, which is the quotient of X by the equivalence relation identifying two infinitesimally closed points. The result of such a quotient is not a scheme, but it can be described nicely as a sheaf on schemes (see Chapters 4 and 5 of *New Spaces in Physics* and [21, 80]).

[3] Algebraic geometry was able to deal successfully with "multiple points with complex coordinates at infinity"; derived algebraic geometry added to these features the possibility to work with quotients by nonfree group actions and self-intersection of such points.

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not really evolved into more sophisticated types of spaces (incorporating singularities and points with symmetries). The recent rise of *derived analytic geometry* might change this [59, 70, 71].

Algebraic geometry depends on the existence of a well-defined dictionary between the geometric features of affine schemes and the algebraic features of commutative rings. This successful translation has led to several attempts to generalize it for other kinds of algebraic structures. The most famous attempt is given by the geometry of noncommutative rings. The attempts to build an actual topological space (a spectrum) from noncommutative rings have not been entirely satisfactory [74, 91],[4] but the dual attempt to characterize geometric features in noncommutative terms has had more success (see Chapter 10 and Chapter 1 of the companion volume, and references therein). However, some important geometric notions are absent from both these approaches (e.g., open subsets, étale maps, the local/global dialectic), preventing a geometric intuition of noncommutative features in classical terms. Other offsprings of algebraic geometry have been *relative geometry*, which develops a geometry for various contexts of commutative monoids (see Chapter 7 and [88]), the geometry of *Berkovich spaces* dual to non-Archimedean fields [9, 12], the *tropical geometry* dual to tropical semirings [35, 66], and the conjectural *geometry over the field with one element* [19, 27, 82].

2.3 Topology

The notion of topological space has been robust enough to successfully deal with some of the new spaces invented in the second half of the 20th century, such as fractals, strange attractors, and nonseparated spaces (such as the Zariski spectrum of a commutative ring). Even the study of topological spaces by means of rings of continuous functions (motivated by Stone and Gelfand dualities) has not introduced new objects [32].

Nonetheless, new spaces have been invented for the needs of topology. For example, the close relationship between topology and intuitionist logic à la Heyting has led to *locale theory*, a variation on topological spaces well suited to define interpretations of logical theories (see Chapter 4 and [45, 92]). Also, in algebraic geometry, the remarkable analogy between Galois theory of fields and the theory of covering spaces [26] has motivated the search for a functor associating a topological space to a commutative ring (a spectrum) that could transport, so to speak, the former theory into the latter. The Zariski spectrum fails to satisfy this, and the proper answer was found with the *étale*

[4] Mostly by lack of functoriality of the spectra.

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spectrum. However, étale spectra could no longer be defined as topological spaces anymore but rather were defined as topoi [5, 23]. Essentially, a *topos* is a new kind of space defined by its category of sheaves instead of its poset of open subspaces. This broader definition led to many new topological objects that are not topological spaces (see Chapter 4 and [46, 64]).

Another motivation for enhancing the notion of topological space was the study of badly separated spaces [2], for example, spaces that have many points but a trivial topology, such as the irrational torus $\mathbb{T}_\alpha := \mathbb{R}/(\mathbb{Z} \oplus \alpha\mathbb{Z})$ ($\alpha \notin \mathbb{Q}$), the leaf spaces of foliations with dense leaves, or even bizarre quotients like $\mathbb{R}/\mathbb{R}_{dis}$ (the continuous $\mathbb{R}$ quotiented by the discrete $\mathbb{R}$). The theory of topoi turned out to be well suited to studying these spaces.[5] But other methods have been developed, like topological sheaves and stacks (inspired by algebraic geometry) [10, 17, 20] or noncommutative geometry à la Connes (see Chapter 1 of the companion volume, and references therein), diffeologies (see Chapter 1, and references therein), or orbifolds and Lie groupoids [56, 67, 75, 90].

2.4 Differential Geometry

Differential geometry has not escaped the development of new types of spaces, but the size of the field has perhaps kept most of it within the classical paradigm. From Riemannian geometry to knot theory, the basic notion is still that of the manifold. Overall, the field does not seem to be in a hurry to incorporate the developments of algebraic geometry (e.g., duality algebra/geometry, singular spaces, functorial approach to moduli spaces and infinite dimensions, relativization with respect to a base space, tangent complexes). Many attempts have been made to improve manifolds, but none of them seems to have become central. An example is *diffeology* theory, which provides a nice framework to deal with infinite-dimensional spaces as well as quotients (see Chapter 1, and references therein). Another one is *synthetic differential geometry*, which enhances the notion of the manifold by authorizing singular points and nilpotent coordinates (see Chapter 2, and references therein).[6] Related approaches have tried to ground differential geometry in the algebraic notion of the C^∞-ring [48, 68, 69]. The most

[5] They are called *étendues* in topos theory; see [5, 46].
[6] Synthetic differential geometry, as its name suggests, also promulgates an axiomatic approach to geometry.

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successful new notion of differentiable space is perhaps that of *orbifolds*, motivated among other things by Thurston's geometrization program [75, 85]. Orbifolds have brought to the field some tools from higher category theory like stacks [56, 67] and equivariant homotopy theory [76].

Another domain using such methods is *microlocal analysis*, where sheaves and their derived categories are of great help for dealing with the problem of extending local solutions of differential equations (see Chapter 3, and references therein).

The most impressive display of postclassical methods in differential geometry can be found in *symplectic geometry* (together with the related fields of *Poisson* and *contact geometries*). Symplectic geometry is a contemporary descendent of analytical mechanics. The notions of symplectic manifold and their Lagrangian submanifolds have given a new geometrical meaning to many constructions of mechanics (e.g., extremal principles and generating functions, covariant phase spaces, Nœther symmetries and reduction [40, 50, 83]). A central operation in the theory is *symplectic reduction*, which combines the restriction to a subspace of a symplectic manifold with a group quotient [65]. Since these two operations might create singularities, symplectic geometry has been forced to deal with both nontransverse intersections and quotients of non-free group actions. These issues have led to the use of new formalisms, such as cohomological methods [41, 54], Lie groupoids and stacks [95, 96], and, eventually, derived geometry (see Chapter 4 of the companion volume and [14, 89]). Also, the application of symplectic geometry to physics has imported many methods from higher category theory: cohomological methods in deformation quantization [15, 53], Fukaya categories in mirror symmetry [52, 78], and, more recently, a whole new interpretation of gauge theory in terms of stacks (see Chapter 5 of the companion volume, and references therein). In fact, more than a simpler user of higher categories and derived algebraic geometry, symplectic geometry has been an important catalyzer in the development of these theories.

Another important innovation with respect to the notion of manifold has been the interpretation of manifolds with boundaries and cobordisms in terms of *higher categories with duals*, a viewpoint that was inspired by topological field theories in physics [6, 7, 60]. In the same way that homotopy theory has transformed topological spaces into tools that can be used to work with ∞-groupoids, this view on cobordisms does not address manifolds as an object on its own but rather as a tool to encode the combinatorial structure of some higher categories.

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2.5 Conclusion

We have referred to the understanding of the notion of space in the middle of the 20th century as the "classical paradigm." This raises the question of whether the many evolutions undergone by the notion of space since then qualify as a new paradigm. As the previous presentation and Table 0.3 illustrate, the classical geography of mathematical spaces is still pertinent today. The conceptual categories organizing the intuition of space have not fundamentally changed (points, functions, local/global dialectic, etc.). If something radical has changed, it will not be found there.

In our opinion, the most important postclassical change has in fact not concerned spaces directly – although it had a tremendous impact on them – but *sets*. If there has been a paradigm shift in mathematics, it has been the enhancement of *set theory* in *category theory* (in which we include higher categories). Category theory is responsible for most of the new spatial features:

1. The most important change has been that sets of points have been enhanced in categories of points (in particular, points can have symmetries).
2. The definition of a space by means of a poset of open subsets has been enhanced in a definition by means of categories of sheaves (topoi, dg-categories, stable categories, etc.).
3. Functions with values in set-based objects (numbers, manifolds, etc.) have been enhanced by functions with values in category-based objects (stacks, moduli spaces, etc.).
4. Many spaces are defined as functors (schemes, moduli spaces, stacks, diffeologies, etc.).
5. Homotopy types are now seen as ∞-groupoids.
6. Also, the relation with logic and axiomatization is made by means of categorical semantics for logical theories.

In the classical paradigm, sets can be thought of as the most primitive notion of space – collecting things together in a minimalist way – from which other notions of space are formally derived. In the new paradigm, categories, and particularly higher categories, are the new primitive spatial notion from which the others are derived. Nowadays, categories are everywhere in topology and geometry, from the definition of the basic objects to the problems and methods of study. The reader will realize that category theory is central in *all* the chapters of this volume.

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