Introduction

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New Spaces in Physics

Two fundamental scientific revolutions took place in physics during the first decades of the 20th century: Einstein's geometric description of the gravitational interaction by means of the *general theory of relativity* and the development of *quantum mechanics*. These two revolutions radically modified our understanding of the laws that rule the physical phenomena taking place at opposite (astrophysical and microscopic) spatiotemporal scales.

On one hand, general relativity introduced into physics essential geometric ideas and tools mainly developed during the 19th century in pure mathematics, notably differential geometry, Riemannian geometry, and tensor calculus. Moreover, general relativity provided the motivating example of the general program – launched by H. Weyl around 1918 – intended to provide similar geometric descriptions of the other fundamental (electromagnetic and nuclear) interactions. This "geometrization program" was finally achieved in the 1950s in the framework of the *Yang–Mills theories* and acquired a solid mathematical foundation and geometric interpretation with the theory of Cartan (for general relativity) and Ehresmann (for Yang–Mills theories) connections on principal

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fiber bundles.[1] Both general relativity and Yang–Mills theories define the so-called *gauge theories of fundamental interactions*, where the term *gauge* refers to the fact that these theories are endowed with a *local symmetry* associated to the possibility to choose different coordinate systems ("gauges") at each spatiotemporal location. Moreover, this geometrization of the fundamental interactions provided the cornerstone of the so-called *standard model* of elementary particles and the associated attempts to unify the four fundamental interactions (where the most celebrated success of this program up to now was the Glashow–Weinberg–Salam unification of the electromagnetic and weak interactions).

On the other hand, quantum mechanics – with its utilization of noncommutative operator algebras on Hilbert spaces – has a strong algebraic flavor that has obstructed to a certain extent the construction of a conceptual interpretation based on a geometric intuition. The main obstacle to the comprehension of quantum mechanics in geometric terms is given by the *noncommutative* character of the algebras of quantum observables. Indeed, this central feature of the quantum formalism has as a consequence that – differently from the *commutative* algebras of classical observables – the quantum observables cannot be understood as functions on an "ordinary" space. This essential feature of quantum mechanics introduces a sort of discontinuity between this theory on one hand and both classical mechanics (which relies on a solid geometric intuition) and the gauge theories of the fundamental interactions on the other.

Roughly speaking, the main lines of research leading to new notions of space in physics after the quantum and the relativistic revolutions can be understood as attempts to understand quantum mechanics in more geometric terms on one hand and to quantize general relativity on the other. Let us consider first the "geometrization" of quantum mechanics. Is it possible to construct noncommutative quantum algebras out of geometric structures? What would be gained by doing so? First, it is worth stressing that quantum mechanics is a formalism that – up to now – could not be endowed with a unanimously accepted conceptual interpretation, being the landscape of competing interpretations populated with radically different conceptual schemes. Now, casting quantum mechanics in more geometric terms redounds in a gain of a conceptual and more intuitive understanding that might pave the road for solving this interpretative conundrum. For instance, the *geometric quantization* formalism developed by Kirillov, Kostant, and Souriau presents quantum mechanics in the same geometric formalism – the theory of connections on fiber bundles – used in gauge theories (see [5, 18, 19, 31]). Now, since gauge

[1] For a history of the path that led from general relativity to Yang–Mills theories (and a collection of some of the corresponding seminal papers), see [27].

theories are better understood than quantum mechanics from a conceptual standpoint, geometric quantization provides a useful bridge to transport this conceptual clarity to quantum mechanics. Second, since classical mechanics relies on a clear geometric basis, the geometrization of quantum mechanics might improve the comprehension of the relationship between quantum mechanics and classical mechanics.

Among the different ways according to which mathematicians can construct noncommutative algebras from geometry, three constructions became relevant in physics, namely,

- the deformation of a ring of functions (giving rise in particular to the *deformation quantization* of a Poisson manifold; see, for instance, [32] and references therein);
- the endomorphisms of a fiber bundle (giving rise in particular to the *geometric quantization* of a symplectic manifold [5, 18, 19, 31]);
- the convolution algebra of a groupoid (giving rise to *noncommutative methods* [6]).

Now, both deformation quantization and geometric quantization strongly rely on the *symplectic formulation of classical mechanics*. Here a main player for the development of physical geometry during the 20th century enters the scene: *symplectic geometry*. Thanks to the work of mathematicians like Arnold, Maslov, Souriau, and Weinstein, among others, the explosion of research in symplectic geometry during the 20th century led to a deep transformation of our comprehension of classical mechanics (see, for instance, [1, 2, 10, 21, 22, 31]). In the framework of this symplectic geometrization of classical mechanics, fundamental new notions and theories were introduced, such as Souriau's *moment map* [25, 31], the Marsden–Weinstein *symplectic reduction* [23], and Weinstein's *symplectic "category"* and *Lagrangian correspondences* [33]. In the wake of this symplectic refoundation of classical mechanics, it is also worth mentioning the development of the theory of variational calculus on jet bundles and the development of multisymplectic geometry launched by De Donder and Weyl and continued – more recently – by Kijowski, among others. In this extended context, important new notions were introduced, such as the *covariant phase space*, the *Peierls bracket*, and the *variational bicomplex* (see, for instance, [7, 8, 17, 26, 31, 34]).

From a conceptual standpoint, the great importance of symplectic (and Poisson) geometry is that it encodes what we could call the *classical seeds* of quantum mechanics. By doing so, the development of symplectic geometry allowed us to significantly reduce the gap between classical and quantum mechanics. It could even be argued that symplectic geometry opened the path to the comprehension of quantum mechanics as a continuous extension of classical mechanics and no longer as a sort of "new paradigm" discontinuously separated from the classical one (see Schreiber's contribution in Chapter 5). For instance, both in *deformation quantization* and in *geometric quantization*, classical structures (namely, the Poisson structure and the symplectic structure, respectively) encode fundamental quantum features. While in deformation quantization the Poisson structure provides the first term of the "quantum" deformation (in the formal parameter $\hbar$) of the commutative algebra of functions on a phase space, in geometric quantization the symplectic structure defines the curvature (on the prequantization fiber bundle) that explains the noncommutativity of quantum operators.[2] Moreover, one of the central facts of symplectic geometry is the existence of a correspondence defined by the symplectic structure between observables (functions on a phase space) and what could be called *classical operators* (Hamiltonian vector fields). In this way, the fundamental role played by operators in mechanics – far from being a quantum innovation – is already a central feature of classical mechanics.[3] It is also worth mentioning that the category-theoretic "points" of a symplectic manifold are given by its Lagrangian submanifolds.[4] According to Guillemin and Sternberg, the notion of Lagrangian submanifold encodes the classical seeds of the quantum indeterminacies:

> *The Heisenberg uncertainty principle says that it is impossible to determine simultaneously the position and momentum of a quantum-mechanical particle. This can be rephrased as follows: the smallest subsets of classical phase space in which the presence of a quantum-mechanical particle can be detected are its Lagrangian submanifolds. For this reason it makes sense to regard the Lagrangian submanifolds of phase space* [rather than its set-theoretic points] *as being its true 'points'* [11].

In this way, it could be argued that if the notion of localization in phase space (in the sense of "being at a certain point" of phase space) is not defined with respect to its set-theoretic points but rather with respect to the Lagrangian "points," then the Heisenberg indeterminacy principle does not forbid a localization of a quantum particle in phase space. All in all, these

[2] It is worth noting that this is in complete analogy to the fact that in general relativity and Yang–Mills theories, the noncommutativity of parallel transports results from the presence of a nontrivial curvature.
[3] In the framework of geometric quantization, quantum operators are in fact defined by means of a vertical extension (where *vertical* means in the direction of the fibers of the corresponding prequantization fiber bundle) of these classical operators (see, for instance, [5]).
[4] Considered from the standpoint of category theory, the Lagrangian submanifolds of a symplectic manifold (M,ω) are the $(*,0)$-points of M in Weinstein's symplectic "category" (where $(*,0)$ is the trivial symplectic manifold), that is, the morphisms (Lagrangian correspondences) $(*,0) \to (M,\omega)$.

different insights brought forward by the development of symplectic geometry are permitting us to progressively sublate the simplistic opposition between the supposedly stable and well-understood realm of classical mechanics and the still-unsolved conceptual problems posited by quantum mechanics. By pushing this line of thought to its limit, it could even be argued that the missing insights permitting us to construct a satisfactory conceptual interpretation of quantum mechanics might stem from a better comprehension of classical mechanics and its symplectic foundations. In this sense, the explosion of research in symplectic geometry is pulling back the problem of interpreting quantum mechanics to an unexpected problem: the problem of reinterpreting classical mechanics.

Another direct repercussion on geometry elicited by the development of quantum mechanics is given by the study of hypothetical "spaces" supporting (or dual to) *noncommutative* "algebras of functions." The new branch of geometry known as *noncommutative geometry* might have been inspired by the capacity to generate new notions of space associated to the *geometry–algebra dualities*, that is, to the dualities between spaces and the algebras of "functions" on them (for instance, the duality between affine schemes and commutative rings or the Gelfand–Naimark duality between compact Hausdorff topological spaces and commutative unital C^*-algebras). Indeed, the geometry–algebra dualities naturally lead to the introduction of new spaces by means of the following pattern: given a particular instantiation of a geometry–algebra duality, one can generalize the corresponding algebra of functions – by passing, for instance, to noncommutative algebras – and try to interpret the new algebra as an "algebra of functions" on a generalized space. However, it is not clear to what extent the noncommutative approaches to geometry do really produce "noncommutative spaces" dual to the corresponding algebras. An alternative way to understand noncommutative geometry could be the following. Given "ordinary" (commutative) spaces, one can define noncommutative invariants. These invariants do not always allow us to reconstruct the space, but they encode nonetheless certain important geometric aspects like *properness* or *smoothness* (see, for instance, Chapter 10 of *New Spaces in Mathematics*). The important fact is that these noncommutative invariants endowed with a geometric meaning permit us to introduce certain geometric concepts and intuitions into the realm of noncommutative algebra.

The formulation of quantum mechanics and general relativity naturally leads to the *quantum gravity* program, that is, to the different research programs intended to quantize general relativity (for instance, superstring theory, loop quantum gravity, semiclassical quantum gravity, causal sets, dynamical triangulations, lattice quantum gravity, and the asymptotic safety program, among others[5]). The general expression *quantize general relativity* denotes here both the application of standard quantization methods (e.g., canonical quantization, path integral) to general relativity in its Lagrangian or Hamiltonian formulation and the direct construction of a theory out of which general relativity and the continuum description of spacetime is supposed to emerge in some "classical" approximation.

The supposed necessity to quantize general relativity can be justified on different grounds, for instance,

- the idea that quantum gravity is required to deal with spacetime singularities taking place at very high energies and very small scales (such as the big bang and black hole singularities);
- the fact that while general relativity describes (by means of the Einstein field equations) the coupling between *classical* matter and the geometry of spacetime, all matter is currently described in the framework of *quantum* field theory;
- the idea that the unification between gravity and the other *quantum* gauge fields carrying the electromagnetic and nuclear interactions requires us also to describe gravity in quantum terms – by taking into account that the nongravitational interactions are mediated by the so-called *gauge bosons* (like the photon for the electromagnetic interaction), this argumentative line led (mainly in the framework of perturbative string theory) to the postulation of a hypothetical massless spin-2 particle that mediates the gravitational interaction, the *graviton*;[6]
- the arguments based on the finite character of black hole entropy (see, for instance, [29]).

Besides these particular motivations, a more straightforward argument is the following. Since

1. general relativity is already a *classical* theory in the sense that it can be cast in terms of classical (Hamiltonian or Lagrangian) mechanics (e.g., ADM formalism, Einstein–Hilbert action); and
2. classical mechanics has been superseded by (or extended to) quantum mechanics,

then general relativity has to be recast in quantum-mechanical terms.

[5] For an overview of different approaches to quantum gravity, see, for instance, [24] and references therein.
[6] It is worth noting that a straightforward application of the perturbative methods of quantum field theory to the gravitational interaction leads to a perturbative nonrenormalizability. This obstacle has been the main motivation for the development of *nonperturbative approaches* to quantum gravity.

Despite the still highly speculative nature of the field, research in quantum gravity has already had a significant impact on mathematical geometry. First, string theory already had important repercussions on research in pure geometry (e.g., mirror symmetry, Gromov–Witten invariants, and enumerative geometry; see, for instance, [3, 15, 16]). Second, research in quantum gravity opened the field of *quantum geometry*, that is, the study of different geometric structures, out of which the classical and continuum spacetime geometry described by general relativity can be reobtained in some form of "classical" limit. In very general terms, the field of quantum geometry explores ideas such as

- a fundamental discretization of spacetime (an idea that goes back to Riemann [28] and reappears in almost every approach to quantum gravity);
- spaces described by noncommutative coordinates (e.g., noncommutative geometry);
- quantum indeterminacies and fluctuations of geometric quantities;
- linear superpositions of geometries.

For instance (as Mariño explains in Chapter 9), string theory addresses different forms of deformation (stringy, quantum) of classical Riemannian geometry resulting from the quantum description of dynamical extended objects (strings and eventually p-branes). In turn, loop quantum gravity studies certain geometric structures – the canonical spin-networks and the covariant spinfoams – arising from a more or less direct quantization of general relativity (see Han's contribution in Chapter 8). Other approaches explore the possibility of understanding the classical and continuum description of space-time geometry – as well as geometric notions like *dimension* and *locality* – as an emergent description arising from *nongeometric* or *pregeometric* (a term introduced by Wheeler [34]) degrees of freedom. Examples of these supposed pregeometric structures are the *causal sets*, that is, sets representing spacetime events endowed with an order relation encoding the causal structure [9, 30], or combinatorial structures like simplicial objects and graphs (e.g., *quantum graphity* [14]). However, the characterization of these structures as non- or pregeometric is problematic (do they really "*break loose at the start from all mention of geometry and distance*?" [34]), and it might seem more appropriate to state that the different "pregeometric" scenarios proposed thus far remove certain geometric features of the classical and continuum description of spacetime conveyed by general relativity (e.g., continuity, differential structure, distance, dimensionality, or locality).

Let us consider now in some detail the different chapters of this volume.

1 Summaries of the Chapters

1.1 Part I Noncommutative and Supercommutative Geometries

1.1.1 Noncommutative Geometry, the Spectral Standpoint (Alain Connes)

The construction of quotients of spaces has been an important source of definitions of new notions of space. The space of leaves of a dense foliation does not have enough open subsets to be described as a manifold or even as a topological space. The spaces of orbits of group actions that are not free have singularities that a topology or a differential structure cannot encode. Several methods have been invented to work with these objects, some using category theory (e.g., sheaves and stacks, topoi, diffeologies), others algebra. The noncommutative geometry of A. Connes belongs to this latter class. The basic idea is to replace the commutative ring of observable functions on the quotient by the noncommutative convolution algebra of the foliation or the group action. This construction is justified by the fact that, when the quotient exists, the categories of modules over the function ring or over the convolution algebra coincide.[7] However, the latter construction is better behaved than the former.

From a more conceptual standpoint, the basic principle of Connes's noncommutative geometry is to substitute the equivalence relation associated to a quotient operation by the corresponding *action groupoid* of identifications. The main difference between an equivalence relation and a groupoid is that the latter keeps track of the fact that different points might be identified in many different ways (which includes *a fortiori* the particular case of possible nontrivial stabilizers). In this sense, an equivalence relation can be understood as a truncated groupoid where the possibly multiple concrete identifications between two elements are collapsed to the abstract fact that they are equivalent. This transition from equivalence relations to groupoids leads to the consideration of a particular noncommutative algebraic structure, namely, the *convolution algebra on the action groupoid* (where the noncommutativity is a direct consequence of the noncommutativity of compositions in the groupoid). As it was stressed by Connes in [6, §1.1, pp. 40–45], this kind of noncommutative algebra was implicitly discovered by Heisenberg in the seminal 1925 article in which he proposed the matrix formulation of quantum mechanics [12].[8]

[7] Technically, they are Morita-equivalent algebras.
[8] In Heisenberg's matrix formulation, the relations between physical quantities are governed by the noncommutative algebra of matrices that represent these quantities. Connes argued that the Ritz–Rydberg combination principle that models the experimental results provided by atomic

Noncommutative geometry consists in defining a certain number of geometric notions (infinitesimal calculus, integration and measure theory, metric, etc.) in terms of algebras that are not necessarily commutative. The central notion is that of *spectral triplets* $(\mathcal{A}, \mathcal{H}, D)$ encoding a "space" with a metric and a measure theory.[9] The commutative algebra of functions on a Riemannian (spin^c) manifold is reinterpreted by Connes as an algebra of operators acting on a Hilbert space of spinors, and the inverse line element of the Riemannian structure is encoded (in Connes's *distance formula*) by the corresponding Dirac operator. Now, the central insight is that this setting remains valid when we substitute the commutative algebra functions by a noncommutative algebra of operators acting on a Hilbert space.

It is also worth noting that Connes's version of noncommutative geometry is motivated by the problem of quantizing gravity and unifying the four fundamental interactions. The inverse line element defined by the Dirac operator D encodes not only the gravitational interaction (associated as usual to the metric) but also the electromagnetic, and nuclear – weak and strong – interactions (which are associated to the inner fluctuations of the metric). This results in a successful derivation of the Lagrangian of the standard model from a Lorentzian spacetime crossed with a specific finite noncommutative space. Interestingly enough, the different physical forces are unified by means of the metric structure of the noncommutative space, thereby giving rise to a sort of generalized gravity theory.

1.1.2 The Logic of Quantum Mechanics (Revisited) (Klaas Landsman)

Landsman's contribution can be inscribed among the attempts to generalize the classical notions of space by using the framework provided by the geometry–algebra duality. Starting with

- the (constructive versions of the) Gelfand–Naimark duality between *commutative* unital C^*-algebras and compact Hausdorff topological spaces; and
- the Stone duality between the category of boolean lattices (with homomorphisms of orthocomplemented lattices as arrows) and totally disconnected compact Hausdorff spaces (Stone spaces),

spectroscopy (which were incompatible with the classical predictions) can be encoded in a groupoid of frequencies whose convolution algebra is nothing but the algebra of matrices discovered by Heisenberg.

9 More precisely, a general spectral triplet $(\mathcal{A}, \mathcal{H}, D)$ is given by a $*$-algebra $\mathcal{A}$ endowed with a representation by bounded operators on a Hilbert space $\mathcal{H}$ and an unbounded self-adjoint Dirac operator D acting on $\mathcal{H}$ and encoding a generalized notion of distance that extends the Riemannian notion of distance to the noncommutative realm.

Landsman moves forward to the intuitionistic/noncommutative realm by addressing

- the Priestley duality between bounded distributive lattices and Priestley spaces; and
- the Esakia duality between Heyting algebras and Esakia spaces.

The ultimate goal of this progression is a conjectured duality between arbitrary unital C^*-algebras and some Heyting algebras. The result of this work in progress would be the construction of a model of an *intuitionistic quantum logic* that has the opposite features from Birkhoff and von Neumann's quantum logic [4]. This means that such an intuitionistic quantum logic is distributive (which paves the way to an interpretation of the logical operations $\wedge$ and $\vee$ as a disjunction and a conjunction, respectively) but does not keep the law of the excluded middle (which, according to Landsman, matches quantum features such as Schrödinger cat situations).

Interestingly enough, this construction of an intuitionistic quantum logic can be related to topos theory. Briefly, we can associate to any unital C^*-algebra A the topos of covariant functors $\mathcal{C}(A) \to Set$ on the posetal category $\mathcal{C}(A)$ of all unital commutative subalgebras of A.

1.1.3 Supergeometry in Mathematics and Physics (Mikhail Kapranov)

Kapranov's contribution addresses the quandaries of supergeometry in mathematics and supersymmetry in physics from an original homotopical perspective. According to Kapranov, the challenge posited by supergeometry and supersymmetry is to understand the formal and conceptual structures underlying the $\pm$ sign rules that govern the supercommutation structures in both mathematics and physics. These structures involve vector spaces with a $\mathbb{Z}/2\mathbb{Z}$-grading together with a monoidal structure involving Koszul's sign rule. Now, an important caveat is here necessary: the similarities between formalisms discovered by physicists and mathematicians might sometimes be misleading. According to Kapranov, an instance of this danger is provided by these supercommutative structures. Indeed, a careful comparative study of supercommutative structures in mathematics and physics leads Kapranov to conclude that the formal similarity should not lead to an identification: the $\mathbb{Z}/2\mathbb{Z}$ of mathematicians is not the same as the $\mathbb{Z}/2\mathbb{Z}$ of physicists.

From a mathematical standpoint, supergeometry is the study of geometric objects whose rings of functions are commutative superalgebras $A = A^{\overline{0}} \oplus A^{\overline{1}}$ composed of even and odd elements subjected to the corresponding supercommutation rules. In this way, supergeometry can be added to the list of