# THE CHARACTER THEORY OF FINITE GROUPS OF LIE TYPE

Through the fundamental work of Deligne and Lusztig in the 1970s, further developed mainly by Lusztig, the character theory of reductive groups over finite fields has grown into a rich and vast area of mathematics. It incorporates tools and methods from algebraic geometry, topology, combinatorics and computer algebra, and has since evolved substantially.

With this book, the authors meet the need for a contemporary treatment, complementing in core areas the well-established books of Carter and Digne–Michel. Focusing on applications in finite group theory, the authors gather previously scattered results and allow the reader to get to grips with the large body of literature available on the subject, covering topics such as regular embeddings, the Jordan decomposition of characters, *d*-Harish-Chandra theory and Lusztig induction for unipotent characters.

Requiring only a modest background in algebraic geometry, this useful reference is suitable for beginning graduate students as well as researchers.

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# The Character Theory of Finite Groups of Lie Type

## A Guided Tour

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# Preface

The subject of this book is the character theory of finite groups of Lie type (or finite reductive groups), following the geometric approach initiated by the fundamental work of Deligne and Lusztig [Lu75], [DeLu76] in the 1970s. Since then, and to its full extent mainly by the monumental work of Lusztig, this has grown into an extremely rich, complex and vast theory, incorporating tools and methods from algebraic geometry, topology, combinatorics and computer algebra. (Lusztig's papers since 1975 on this subject alone comprise already a few thousand densely written pages.) One of the ultimate aims of this theory is to reduce the computation of character tables of whole series of finite groups of Lie type (e.g., the series of groups $E_8(q)$ where $q$ is any prime power) to purely combinatorial tasks that could, for example, be performed automatically on a computer. For the general linear groups $\mathrm{GL}_n(q)$ this was already achieved in principle by Green [Gre55] in 1955, but the analogous problem for the closely related special linear and unitary groups $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$ is still not completely solved.

Within finite group theory, the importance of this subject is highlighted by the classification of finite simple groups: apart from the alternating groups and the 26 sporadic simple groups, all non-abelian finite simple groups are 'of Lie type'. According to Aschbacher [Asch00], [Asch04] when faced with a problem about finite groups, it nowadays seems best to attempt to reduce the problem or a related problem to a question about simple groups or groups closely related to simple groups. The classification then supplies an explicit list of groups which can be studied in detail using the effective description of the groups. In recent years, this programme has led to substantial advances on various long-standing open problems in the representation theory of finite groups: these were shown to reduce to questions on simple groups which could then be solved by applying the deep results on characters of finite reductive groups; see, for example, the book of Navarro [Na18] and the second author's survey [Ma17].

Lusztig's book [Lu84a] is a milestone in the study of representations of finite groups of Lie type, both in terms of conceptual depth and technical complexity. It brings together various deep and rich theories, culminating in the fundamental 'Jordan decomposition of characters'. In particular, this provides a classification of the irreducible characters, and formulae for character degrees, in terms of purely combinatorial data.

The books by Carter [Ca85] and Digne–Michel [DiMi20] provide more background material and have become influential and highly useful references in this area. Further, more recent texts dealing with more specific aspects are the books by Bonnafé [Bo06] and by Cabanes–Enguehard [CE04]. In our book, the primary focus is on explaining Lusztig's classification of the irreducible characters, and surrounding topics like complete root data, regular embeddings, degree and character formulae, Lusztig induction and restriction and Jordan decomposition. Thus, we will complement, enhance and go beyond the above texts in several ways:

- A substantial part of Chapter 1 is concerned with the discussion of explicit constructions involving root data and Steinberg maps; these are at the basis of efficient algorithms and computer implementations for which the CHEVIE system [GHLMP], [MiChv] is our primary reference.
- Chapter 2 provides an introduction (with complete proofs where possible) into the basic formalism of Lusztig's book [Lu84a] leading to the statement of the 'Jordan decomposition of characters', both in the 'connected centre' case and in general. We also discuss the computation of Green functions, a problem which is not yet solved in complete generality.
- In Chapter 3 we present not only the well-established ordinary Harish-Chandra theory but also $d$-Harish-Chandra theories defined by means of Lusztig induction, which have proved to be of fundamental importance in block theory of finite reductive groups.
- The classification of the all-important unipotent characters of finite reductive groups and various of their properties are described in Chapter 4, including a discussion of Lusztig's 'non-abelian Fourier transforms'. We also discuss the decomposition of the Lusztig functor and its commutation with Jordan decomposition for groups with connected centre. Furthermore, we touch upon some topics in the character theory of disconnected reductive groups.
- An appendix discusses, in a somewhat informal way, various applications, open problems and connections to related theories, with numerous references to further reading.

Throughout, we have tried to design the exposition of the above material with a view towards applications in finite group theory, and to be accessible to a reader with only a modest background in algebraic geometry. In view of the enormous amount of material available in this area, it is clear that we had to make a number of choices concerning the topics that we cover in this book. For example, we have decided not to say anything about the classification of the conjugacy classes of finite groups of Lie type (except for some occasional general statements). Furthermore, as a general rule, we only give proofs for statements for which we could not find any convenient reference in the existing literature. (But sometimes we defer from this rule and give a detailed argument when this appears to be a good illustration for the methods developed so far.)

On the other hand, we have made a serious attempt to provide precise references, thereby giving something like a guided tour through this vast territory. In short, we hope that this text will be a useful addition to the literature on the character theory of finite groups of Lie type, where the choice of topics and the style of exposition have been strongly influenced, of course, by our own experience with the sometimes difficult task of finding appropriate references, or accommodating the existing literature to specific needs in applications.

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