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Reductive Groups and Steinberg Maps

This first chapter is of a preparatory nature; its purpose is to collect some basic results about algebraic groups (with proofs where appropriate) which will be needed for the discussion of characters and applications in later chapters. In particular, one of our aims is to arrive at the point where we can give a precise definition of a 'series of finite groups of Lie type' $\{\mathbf{G}(q)\}$, indexed by a parameter q. We also introduce a number of tools which will be helpful in the discussion of examples.

For a reader familiar with the basic notions about algebraic groups, root data and Frobenius maps, it might just be sufficient to browse through this chapter on a first reading, in order to see some of our notation. There are, however, a few topics and results that are frequently used in the literature on algebraic groups and finite groups of Lie type, but for which we have found the coverage in standard reference texts (like [Bor91], [Ca85], [DiMi20], [Hum91], [Spr98]) not to be sufficient; these will be treated here in a fairly self-contained manner.

Section 1.1 is purely expository: it introduces affine varieties, linear algebraic groups in general, and the first definitions concerning reductive algebraic groups. In Section 1.2, we consider in some detail (abstract) root data, the basic underlying combinatorial structure of the theory of reductive algebraic groups. We present an approach (familiar in the literature concerned with computational aspects, e.g. [CMT04], [BrLu12]) in which root data simply appear as factorisations of the Cartan matrix of a root system. This will be extremely useful for the discussion of examples and the efficient construction of root data from Cartan matrices.

Section 1.3 contains the fundamental existence and isomorphism theorems of Chevalley [Ch55], [Ch05] concerning connected reductive algebraic groups. We also state the more general 'isogeny theorem' and present some of its basic applications. (There is now a quite short proof available, due to Steinberg [St99].) An important class of homomorphisms of algebraic groups to which this more general theorem applies are the Steinberg maps, to be discussed in detail in Section 1.4.

Following [St68], one might just define a Steinberg map of a connected reductive algebraic group **G** to be an endomorphism whose fixed point set is finite. But it will be important and convenient to single out a certain subclass of such morphisms to which one can naturally attach a positive real number q (some power of which is a prime power) and such that one can speak of the corresponding finite group $\mathbf{G}(q)$. The known results on Frobenius and Steinberg maps are somewhat scattered in the literature so we treat this in some detail here, with complete proofs.

In Section 1.5, we illustrate the material developed so far by a number of further basic constructions and examples. In Section 1.6, we show how all this leads to the notion of 'generic' reductive groups, in which q will appear as a formal parameter. Finally, Section 1.7 discusses in some detail the first applications to the character theory of finite groups of Lie type: the 'Multiplicity–Freeness' Theorem 1.7.15.

1.1 Affine Varieties and Algebraic Groups

In this section, we introduce some basic notions concerning affine varieties and algebraic groups. We will do this in a somewhat informal way, assuming that the reader is willing to fill in some details from textbooks like [Bor91], [Ca85], [Ge03a], [Hum91], [MaTe11], [Spr98].

1.1.1 Affine varieties. Let k be a field and let $\mathbf{X}$ be a set. Let A be a subalgebra of the k-algebra $\mathscr{A}(\mathbf{X}, k)$ of all functions $f: \mathbf{X} \to k$. Using A we can try to define a topology on $\mathbf{X}$: a subset $\mathbf{X}' \subseteq \mathbf{X}$ is called closed if there is a subset $S \subseteq A$ such that $\mathbf{X}' = \{x \in \mathbf{X} \mid f(x) = 0 \text{ for all } f \in S\}$. This works well, and gives rise to the *Zariski topology* on $\mathbf{X}$, if A is neither too small nor too big. The precise requirements are (see [Car55-56]):

(1) A is a finitely generated k-algebra and contains the identity of $\mathscr{A}(\mathbf{X}, k)$;
(2) A separates points, that is, given $x \neq x'$ in $\mathbf{X}$, there exist some $f \in A$ such that $f(x) \neq f(x')$;
(3) any k-algebra homomorphism $\lambda: A \to k$ is given by evaluation at a point (that is, there exists some $x \in \mathbf{X}$ such that $\lambda(f) = f(x)$ for all $f \in A$).

A pair $(\mathbf{X}, A)$ satisfying the above conditions will be called an *affine variety over* k; the functions in A are called the *regular functions* on $\mathbf{X}$. We define $\dim \mathbf{X}$ to be the supremum of all $r \geqslant 0$ such that there exist r algebraically independent elements in A. Since A is finitely generated, $\dim \mathbf{X} < \infty$. (See [Ge03a, 1.2.18].) If A is an integral domain, then $\mathbf{X}$ is called *irreducible*.

There is now also a natural notion of morphisms. Let $(\mathbf{X}, A)$ and $(\mathbf{Y}, B)$ be affine varieties over k. A map $\varphi: \mathbf{X} \to \mathbf{Y}$ will be called a *morphism* if composition with φ maps B into A (that is, for all $g \in B$, we have $\varphi^*(g) := g \circ \varphi \in A$); in this case, $\varphi^*\colon B \to A$ is an algebra homomorphism, and every algebra homomorphism $B \to A$ arises in this way. The morphism φ is an *isomorphism* if there is a morphism $\psi\colon \mathbf{Y} \to \mathbf{X}$ such that $\psi \circ \varphi = \mathrm{id}_{\mathbf{X}}$ and $\varphi \circ \psi = \mathrm{id}_{\mathbf{Y}}$. (Equivalently: the induced algebra homomorphism $\varphi^*\colon B \to A$ is an isomorphism.)

Starting with these definitions, the basics of (affine) algebraic geometry are developed in [St74], and this is also the approach taken in [Ge03a]. The link with the more traditional approach via closed subsets in affine space (which, when considered as an algebraic set with the Zariski topology, we denote by $\mathbf{k}^n$) is obtained as follows. Let $(\mathbf{X}, A)$ be an affine variety over k. Choose a set $\{a_1, \ldots, a_n\}$ of algebra generators of A and consider the polynomial ring $k[t_1, \ldots, t_n]$ in n independent indeterminates $t_1, \ldots, t_n$. There is a unique algebra homomorphism $\pi\colon k[t_1, \ldots, t_n] \to A$ such that $\pi(t_i) = a_i$ for $1 \leqslant i \leqslant n$. Then we have a morphism

$$\varphi\colon \mathbf{X} \to \mathbf{k}^n, \qquad x \mapsto (a_1(x), \ldots, a_n(x)),$$

such that $\varphi^* = \pi$. The image of φ is the 'Zariski closed' set of $\mathbf{k}^n$ consisting of all $(x_1, \ldots, x_n) \in \mathbf{k}^n$ such that $f(x_1, \ldots, x_n) = 0$ for all $f \in \ker(\pi)$.

To develop these matters any further, it is then essential to assume that k is algebraically closed, which we will do from now on. One can go a long way towards those parts of the theory that are relevant for algebraic groups, once the following basic result about morphisms is available (see [St74, §1.13], [Ge03a, §2.2]):

Let $\varphi\colon \mathbf{X} \to \mathbf{Y}$ be a morphism between irreducible affine varieties such that $\varphi(\mathbf{X})$ is dense in $\mathbf{Y}$. Then there is a non-empty open subset $\mathbf{V} \subseteq \mathbf{Y}$ such that $\mathbf{V} \subseteq \varphi(\mathbf{X})$ and, for all $y \in \mathbf{V}$, we have $\dim \varphi^{-1}(y) = \dim \mathbf{X} - \dim \mathbf{Y}$.

1.1.2 Algebraic groups. In order to define algebraic groups, we need to know that direct products of affine varieties are again affine varieties. So let $(\mathbf{X}, A)$ and $(\mathbf{Y}, B)$ be affine varieties over k. Given $f \in A$ and $g \in B$, we define the function $f \otimes g\colon \mathbf{X} \times \mathbf{Y} \to k$ by $(x, y) \mapsto f(x)g(y)$. Let $A \otimes B$ be the subspace of $\mathscr{A}(\mathbf{X} \times \mathbf{Y}, k)$ spanned by all $f \otimes g$, where $f \in A$ and $g \in B$. Then $A \otimes B$ is a subalgebra of $\mathscr{A}(\mathbf{X} \times \mathbf{Y}, k)$ (isomorphic to the tensor product of A, B over k) and the pair $(\mathbf{X} \times \mathbf{Y}, A \otimes B)$ is easily seen to be an affine variety over k. Now let $(\mathbf{G}, A)$ be an affine variety and assume that $\mathbf{G}$ is an abstract group where multiplication and inversion are defined by maps $\mu\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and $\iota\colon \mathbf{G} \to \mathbf{G}$. We say that $\mathbf{G}$ is an *affine algebraic group* if μ and ι are morphisms. The first example is the additive group of k which, when considered as an algebraic group, we denote by $\mathbf{k}^+$ (with algebra of regular functions given by the polynomial functions $k \to k$).

Most importantly, the group $\mathrm{GL}_n(k)$ ($n \geqslant 1$), is an affine algebraic group, with algebra of regular functions given as follows. For $1 \leqslant i, j \leqslant n$ let $f_{ij}\colon \mathrm{GL}_n(k) \to k$ be the function that sends a matrix $g \in \mathrm{GL}_n(k)$ to its (i, j)-entry; furthermore, let $\delta\colon \mathrm{GL}_n(k) \to k$, $g \mapsto \det(g)^{-1}$. Then the algebra of regular functions on $\mathrm{GL}_n(k)$

is the subalgebra of $\mathscr{A}(\mathrm{GL}_n(k), k)$ generated by δ and all f_{ij} $(1 \leqslant i, j \leqslant n)$. In particular, $\mathrm{GL}_1(k)$ is an affine algebraic group, which we denote by $\mathbf{k}^\times$.

It is a basic fact that any affine algebraic group $\mathbf{G}$ over k is isomorphic to a closed subgroup of $\mathrm{GL}_n(k)$, for some $n \geqslant 1$; see [Ge03a, 2.4.4]. For this reason, an affine algebraic group is also called a *linear algebraic group*. When we just write 'algebraic group', we always mean a linear algebraic group.

1.1.3 Connected algebraic groups. A topological space is *connected* if it cannot be written as a disjoint union of two non-empty open subsets. A linear algebraic group $\mathbf{G}$ can always be written as the disjoint union of finitely many connected components, where the component containing the identity element is a closed connected normal subgroup of $\mathbf{G}$, denoted by $\mathbf{G}^\circ$; see [Ge03a, 1.3.13]. Thus, $\mathbf{G}$ is connected if and only if $\mathbf{G} = \mathbf{G}^\circ$. (Equivalently: $\mathbf{G}$ is irreducible as an affine variety; see [Ge03a, 1.1.12, 1.3.1].)

What is the significance of this fundamental notion? Every finite group $\mathbf{G}$ can be regarded as a linear algebraic group, with algebra of regular functions given by all of $\mathscr{A}(\mathbf{G}, k)$. Thus, the study of *all* linear algebraic groups is necessarily more complicated than the study of all finite groups. But, as Vogan [Vo07] writes, "a miracle happens" when we consider *connected* algebraic groups: things actually become much less complicated. One reason is that a connected algebraic group is almost completely determined by its Lie algebra (see 1.1.5 and also 1.1.12 below), and the latter can be studied using linear algebra methods.

Combined with our assumption that k is algebraically closed, this gives us some powerful tools. For example, matrices over algebraically closed fields can be put in triangular form. An analogue of this fact for an arbitrary connected algebraic group is the statement that every element is contained in a Borel subgroup (that is, a maximal closed connected solvable subgroup); see [Ge03a, 3.4.9]. A useful criterion for showing the connectedness of a subgroup of $\mathbf{G}$ is as follows.

Let $\{\mathbf{H}_i\}_{i\in I}$ be a family of closed connected subgroups of $\mathbf{G}$. Then the (abstract) subgroup $H = \langle \mathbf{H}_i \mid i \in I\rangle \subseteq \mathbf{G}$ generated by this family is closed and connected; furthermore, $H = \mathbf{H}_{i_1} \cdots \mathbf{H}_{i_n}$ for some n and $i_1, \ldots, i_n \in I$.

The proof uses the result on morphisms mentioned at the end of 1.1.1; see, e.g., [Ge03a, 2.4.6]. Note that, if $\mathbf{U}, \mathbf{V}$ are any closed subgroups of $\mathbf{G}$, then the abstract subgroup $\langle \mathbf{U}, \mathbf{V}\rangle \subseteq \mathbf{G}$ need not even be closed. For example, if $\mathbf{G} = \mathrm{SL}_2(\mathbb{C})$, then it is well known that the subgroup $\mathrm{SL}_2(\mathbb{Z})$ is generated by two elements, of orders 4 and 6, but this subgroup is certainly not closed in $\mathbf{G}$. However, if $\mathbf{V}$ is normalised by $\mathbf{U}$, then $\langle \mathbf{U}, \mathbf{V}\rangle = \mathbf{U}.\mathbf{V}$ is closed; see [Ch05, §3.3, Corollaire].

We will use without further special mention some standard facts (whose proofs also rely on the above-mentioned result on morphisms). For example, if $f : \mathbf{G} \to \mathbf{G}'$ is a homomorphism of linear algebraic groups, then the image $f(\mathbf{G})$ is a closed subgroup of $\mathbf{G}'$ (connected if $\mathbf{G}$ is connected), the kernel of f is a closed subgroup of $\mathbf{G}$ and we have $\dim \mathbf{G} = \dim \ker(f) + \dim f(\mathbf{G})$. (See, e.g., [Ge03a, 2.2.14].)

1.1.4 Classical groups. These form an important class of examples of linear algebraic groups. They are closed subgroups of $\mathrm{GL}_n(k)$ defined by certain quadratic polynomials corresponding to a bilinear or quadratic form on the underlying vector space k^n. There is an extensive literature on these groups; see, e.g., [Bou07], [Dieu74], [Gro02], [Tay92]. Since our base field k is algebraically closed, the general theory simplifies considerably and we only need to consider three classes of groups, leading to the Dynkin types B, C, D. First, and quite generally, for any invertible matrix $Q_n \in M_n(k)$, we obtain a linear algebraic group

$$\Gamma(Q_n, k) := \{A \in M_n(k) \mid A^{\mathrm{tr}} Q_n A = Q_n\};$$

note that $\det(A) = \pm 1$ for all $A \in \Gamma(Q_n, k)$. Let us now take Q_n of the form

$$Q_n = \begin{bmatrix} 0 & \cdots & 0 & \pm 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \pm 1 & \iddots & \vdots \\ \pm 1 & 0 & \cdots & 0 \end{bmatrix} \in M_n(k) \qquad (n \geqslant 2)$$

where the signs are such that $Q_n^{\mathrm{tr}} = \pm Q_n$. Then Q_n is the matrix of a non-degenerate symmetric or alternating bilinear form on k^n; furthermore, $Q_n^{-1} = Q_n^{\mathrm{tr}}$ and $\Gamma(Q_n, k)$ will be invariant under transposing matrices.

If $Q_n^{\mathrm{tr}} = -Q_n$ and n is even, then $\Gamma(Q_n, k)$ will be denoted $\mathrm{Sp}_n(k)$ and called the *symplectic group*. This group is always connected; see [Ge03a, 1.7.4]. Now assume that $Q_n^{\mathrm{tr}} = Q_n$ and that all signs in Q_n are +. Then we also consider the quadratic form on k^n defined by the polynomial

$$f_n := \begin{cases} t_1 t_{2m+1} + t_2 t_{2m} + \ldots + t_m t_{m+2} + t_{m+1}^2 & \text{if } n = 2m + 1 \text{ is odd,} \\ t_1 t_{2m} + t_2 t_{2m-2} + \ldots + t_m t_{m+1} & \text{if } n = 2m \text{ is even,} \end{cases}$$

(where $t_1, \ldots, t_n$ are indeterminates). This defines a function $\dot{f}_n : k^n \to k$, where we regard the elements of k^n as column vectors. Using the notation in [MaTe11, §1.2], the *general orthogonal group* is defined as

$$\mathrm{GO}_n(k) := \{A \in M_n(k) \mid \dot{f}_n(Av) = \dot{f}_n(v) \text{ for all } v \in k^n\};$$

furthermore, $\mathrm{SO}_n(k) := \mathrm{GO}_n^\circ(k)$ will be called the *special orthogonal group*. In each case, we have $[\mathrm{GO}_n(k) : \mathrm{SO}_n(k)] \leqslant 2$; see [Ge03a, §1.7], [Gro02] for further details. Note also that, if $\mathrm{char}(k) \neq 2$, then $\mathrm{GO}_n(k) = \Gamma(Q_n, k)$; furthermore, if n is even and $\mathrm{char}(k) = 2$, then $\mathrm{GO}_n(k)$ will be strictly contained in $\Gamma(Q_n, k)$. (See also Example 1.5.5 for the case where n is odd and $\mathrm{char}(k) = 2$.)

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The particular choices of Q_n and f_n lead to simple descriptions of a BN-pair in $\mathrm{Sp}_n(k)$ and $\mathrm{SO}_n(k)$; see, e.g., [Ge03a, §1.7] (and also 1.1.14 below). The Dynkin types and dimensions are given as follows.

Group	Type	Dimension
$\mathrm{SO}_{2m+1}(k)$	B_m	$2m^2 + m$
$\mathrm{Sp}_{2m}(k)$	C_m	$2m^2 + m$
$\mathrm{SO}_{2m}(k)$	D_m	$2m^2 - m$

Later on, if there is no danger of confusion, we shall just write GL_n, SL_n, Sp_n, SO_n, GO_n instead of $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, $\mathrm{Sp}_n(k)$, $\mathrm{SO}_n(k)$, $\mathrm{GO}_n(k)$, respectively.

1.1.5 Tangent spaces and the Lie algebra. Let $(\mathbf{X}, A)$ be an affine variety over k. Then the *tangent space* $T_x(\mathbf{X})$ of $\mathbf{X}$ at a point $x \in \mathbf{X}$ is the set of all k-linear maps $D\colon A \to k$ such that $D(fg) = f(x)D(g) + g(x)D(f)$. (Such linear maps are called *derivations*.) Clearly, $T_x(\mathbf{X})$ is a subspace of the vector space of all linear maps from A to k. Any $D \in T_x(\mathbf{X})$ is uniquely determined by its values on a set of algebra generators of A. Hence, since A is finitely generated, we have $\dim T_x(\mathbf{X}) < \infty$. If $\mathbf{X}' \subseteq \mathbf{X}$ is a closed subvariety, we have a natural inclusion $T_x(\mathbf{X}') \subseteq T_x(\mathbf{X})$ for any $x \in \mathbf{X}'$. For example, we can identify $T_x(\mathbf{k}^n)$ with k^n for all $x \in \mathbf{k}^n$ and so, if $\mathbf{X} \subseteq \mathbf{k}^n$ is a Zariski closed subset, we have $T_x(\mathbf{X}) \subseteq k^n$ for all $x \in \mathbf{X}$ (see [Ge03a, 1.4.10]). More generally, any morphism $\varphi\colon \mathbf{X} \to \mathbf{Y}$ between affine varieties over k naturally induces a linear map $d_x\varphi\colon T_x(\mathbf{X}) \to T_{\varphi(x)}(\mathbf{Y})$ for any $x \in \mathbf{X}$, called the *differential* of φ at x. (See [Ge03a, §1.4].)

Now let $\mathbf{G}$ be a linear algebraic group and denote $L(\mathbf{G}) := T_1(\mathbf{G})$, the tangent space at the identity element of $\mathbf{G}$. Then

$$L(\mathbf{G}) = L(\mathbf{G}^\circ) \qquad \text{and} \qquad \dim \mathbf{G} = \dim L(\mathbf{G});$$

see [Ge03a, 1.5.2]. Furthermore, there is a Lie product $[\ ,\ ]$ on $L(\mathbf{G})$ which can be defined as follows. Consider a realisation of $\mathbf{G}$ as a closed subgroup of $\mathrm{GL}_n(k)$ for some $n \geqslant 1$. We have a natural isomorphism of $L(\mathrm{GL}_n(k))$ onto $M_n(k)$, the vector space of all $n \times n$-matrices over k; see [Ge03a, 1.4.14]. Hence we obtain an embedding $L(\mathbf{G}) \subseteq M_n(k)$ where $M_n(k)$ is endowed with the usual Lie product $[A, B] = AB - BA$ for $A, B \in M_n(k)$. Then one shows that $[L(\mathbf{G}), L(\mathbf{G})] \subseteq L(\mathbf{G})$ and so $[\ ,\ ]$ restricts to a Lie product on $L(\mathbf{G})$; see [Ge03a, 1.5.3]. (Of course, there is also an intrinsic description of $L(\mathbf{G})$ in terms of the algebra of regular functions on $\mathbf{G}$ which shows, in particular, that the product does not depend on the choice of the realisation of $\mathbf{G}$; see [Ge03a, 1.5.4].)

1.1.6 Quotients. Let $\mathbf{G}$ be a linear algebraic group and $\mathbf{H}$ be a closed normal subgroup of $\mathbf{G}$. We have the abstract factor group $\mathbf{G}/\mathbf{H}$ and we would certainly like to know if this can also be viewed as an algebraic group. More generally, let $\mathbf{X}$ be an affine variety and $\mathbf{H}$ be a linear algebraic group such that we have a morphism $\mathbf{H} \times \mathbf{X} \to \mathbf{X}$ which defines an action of $\mathbf{H}$ on $\mathbf{X}$. The question of whether we can view the set of orbits $\mathbf{X}/\mathbf{H}$ as an algebraic variety leads to 'geometric invariant theory'; in general, these are quite delicate matters. Let us begin by noting that there is a natural candidate for the algebra of functions on the orbit set $\mathbf{X}/\mathbf{H}$: If A is the algebra of regular functions on $\mathbf{X}$, then

$$A^{\mathbf{H}} := \{f \in A \mid f(h.x) = f(x) \text{ for all } h \in \mathbf{H} \text{ and all } x \in \mathbf{X}\}$$

can naturally be regarded as an algebra of k-valued functions on $\mathbf{X}/\mathbf{H}$. However, the three properties in 1.1.1 will not be satisfied in general. There are two particular situations in which this is the case, and these will be sufficient for most parts of this book:

- $\mathbf{H}$ is a finite group, and
- $\mathbf{X} = \mathbf{G}$ is an algebraic group and $\mathbf{H}$ is a closed normal subgroup (acting by left multiplication).

(For the proofs, see [Fo69, 5.25] or [Ge03a, 2.5.12] in the first case, and [Fo69, 2.26] or [Spr98, §5.5] in the second case.) Now let us assume that $(\mathbf{X}/\mathbf{H}, A^{\mathbf{H}})$ is an affine variety. Then, first of all, the natural map $\mathbf{X} \to \mathbf{X}/\mathbf{H}$ is a morphism of affine varieties. Furthermore, we have the following universal property:

If $\varphi\colon \mathbf{X} \to \mathbf{Y}$ *is any morphism of affine varieties that is constant on the orbits of* $\mathbf{H}$ *on* $\mathbf{X}$*, then there is a unique morphism* $\bar{\varphi}\colon \mathbf{X}/\mathbf{H} \to \mathbf{Y}$ *such that* φ *is the composition of* $\bar{\varphi}$ *and the natural map* $\mathbf{X} \to \mathbf{X}/\mathbf{H}$.

(Indeed, if B is the algebra of regular functions on $\mathbf{Y}$, then the induced algebra homomorphism $\varphi^*\colon B \to A$ has image in $A^{\mathbf{H}}$, hence it factors through an algebra homomorphism $\bar{\varphi}^*\colon B \to A^{\mathbf{H}}$ for a unique morphism $\bar{\varphi}\colon \mathbf{X}/\mathbf{H} \to \mathbf{Y}$.)

For example, if we are in the second of the above two cases, then the universal property shows that the induced multiplication and inversion maps on $\mathbf{G}/\mathbf{H}$ are morphisms of affine varieties. Thus, $\mathbf{G}/\mathbf{H}$ is an affine algebraic group.

1.1.7 Algebraic groups in positive characteristic. The finite groups that we shall study in this book are obtained as

$$\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$$

where the $F\colon \mathbf{G} \to \mathbf{G}$ are certain bijective endomorphisms with finitely many fixed points, called '*Steinberg maps*'. (This will be discussed in detail in Section 1.4.) Such maps F will only exist if k has prime characteristic, so we will usually assume that p is a prime number and $k = \overline{\mathbb{F}}_p$ is an algebraic closure of the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Now, algebraic geometry over fields with positive characteristic is, in some respects, more tricky than algebraic geometry over $\mathbb{C}$, say (because of the inseparability of certain field extensions; see also 1.1.8 below). However, some things are actually easier. For example, using an embedding of $\mathbf{G}$ into some $\mathrm{GL}_n(k)$ as in 1.1.2, we see that every element $g \in \mathbf{G}$ has finite order. Thus, we can define g to be *semisimple* if the order of g is prime to p; we define g to be *unipotent* if the order of g is a power of p. Then, clearly, any $g \in \mathbf{G}$ has a unique decomposition

$$g = us = su \qquad \text{where } s \in \mathbf{G} \text{ is semisimple and } u \in \mathbf{G} \text{ is unipotent,}$$

called the *Jordan decomposition of elements*. (In characteristic 0, this certainly requires more work; see [Spr98, §2.4].) Another example: An algebraic group $\mathbf{G}$ is called a *torus* if $\mathbf{G}$ is isomorphic to a direct product of a finite number of copies of $\mathbf{k}^\times$. Then $\mathbf{G}$ is a torus if and only if $\mathbf{G}$ is connected, abelian and consists entirely of elements of order prime to p; see [Ge03a, 3.1.9]. (To formulate this in characteristic 0, one would need the general definition of semisimple elements.)

1.1.8 Some things that go wrong in positive characteristic. Here we collect a few items which show that, when working over $k = \overline{\mathbb{F}}_p$ as above, things may not work as one might hope or expect. First note that a bijective homomorphism of algebraic groups $\varphi\colon \mathbf{G}_1 \to \mathbf{G}_2$ need not be an isomorphism. A standard example is the Frobenius map $\mathbf{k}^+ \to \mathbf{k}^+$, $x \mapsto x^p$. (Note that, over $\mathbb{C}$, a bijective homomorphism between connected algebraic groups is an isomorphism; see [GoWa98, 11.1.16].) In general, we have (see, e.g., [Ge03a, 2.3.15], [Spr98, 5.3.3]):

(a) *A bijective homomorphism of linear algebraic groups* $\varphi\colon \mathbf{G}_1 \to \mathbf{G}_2$ *is an isomorphism if and only if the differential* $d_1\varphi\colon T_1(\mathbf{G}_1) \to T_1(\mathbf{G}_2)$ *between the tangent spaces is an isomorphism.*

The next item concerns the Lie algebra of an algebraic group. Let $\mathbf{G}$ be a linear algebraic group and $\mathbf{U}, \mathbf{H}$ be closed subgroups of $\mathbf{G}$. As already noted in 1.1.5, we have natural inclusions of $L(\mathbf{U})$, $L(\mathbf{H})$ and $L(\mathbf{U} \cap \mathbf{H})$ into $L(\mathbf{G})$. It is always true that $L(\mathbf{U} \cap \mathbf{H}) \subseteq L(\mathbf{U}) \cap L(\mathbf{H})$.

(b) *When considering the intersection of closed subgroups* $\mathbf{U}, \mathbf{H}$ *of an algebraic group* $\mathbf{G}$*, it is not always true that* $L(\mathbf{U} \cap \mathbf{H}) = L(\mathbf{U}) \cap L(\mathbf{H})$.

A good example to keep in mind is as follows. Let $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, $\mathbf{H} = \mathrm{SL}_n(\overline{\mathbb{F}}_p)$ and $\mathbf{Z}$ be the centre of $\mathbf{G}$ (the scalar matrices in $\mathbf{G}$). Then $\mathbf{Z}, \mathbf{H}$ are closed subgroups of $\mathbf{G}$. As in 1.1.5, we can identify $L(\mathbf{G}) = M_n(k)$; then $L(\mathbf{H})$ consists of all matrices of trace 0 and $L(\mathbf{Z})$ consists of all scalar matrices. (For these facts see, for example, [Ge03a, §1.5].) Assume now that p divides n. Then, clearly, $\{0\} \neq L(\mathbf{Z}) \subseteq L(\mathbf{H})$, whereas $\mathbf{Z} \cap \mathbf{H}$ is finite and so $L(\mathbf{Z} \cap \mathbf{H}) = L((\mathbf{Z} \cap \mathbf{H})^\circ) = \{0\}$. (This phenomenon cannot happen in characteristic 0; see [Bor91, 6.12] or [Hum91, 12.5].)

Closely related to the above item is the next item: semidirect products. Let $\mathbf{G}$ be an algebraic group and $\mathbf{U}, \mathbf{N}$ be closed subgroups such that $\mathbf{N}$ is normal, $\mathbf{G} = \mathbf{U}.\mathbf{N}$ and $\mathbf{U} \cap \mathbf{N} = \{1\}$. Following [Bor91, 1.11], we say that $\mathbf{G}$ is the *semidirect product (of algebraic groups)* of $\mathbf{U}, \mathbf{N}$ if the natural map $\mathbf{U} \times \mathbf{N} \to \mathbf{G}$ given by multiplication is an isomorphism of affine varieties. If this holds, we have an inverse isomorphism $\mathbf{G} \to \mathbf{U} \times \mathbf{N}$ and the first projection will induce an isomorphism of algebraic groups $\mathbf{G}/\mathbf{N} \cong \mathbf{U}$. We have the criterion:

(c) *$\mathbf{G}$ is the semidirect product (of algebraic groups) of $\mathbf{U}, \mathbf{N}$ if and only if $L(\mathbf{U}) \cap L(\mathbf{N}) = \{0\}$. For example, this holds if $\mathbf{U}$ or $\mathbf{N}$ is finite.*

(This easily follows from (a) and the description of the differential of the product map $\mathbf{U} \times \mathbf{N} \to \mathbf{G}$; see, e.g., [Ge03a, 1.5.6], [Spr98, 4.4.12].) Take again the above example where $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, $\mathbf{U} = \mathrm{SL}_n(\overline{\mathbb{F}}_p)$ and $\mathbf{N}$ is the centre of $\mathbf{G}$ (the scalar matrices in $\mathbf{G}$). Assume now that $n = p$. Then $\mathbf{U}, \mathbf{N}$ are closed connected normal subgroups such that $\mathbf{G} = \mathbf{U}.\mathbf{N}$ and $\mathbf{U} \cap \mathbf{N} = \{1\}$. However, $\{0\} \neq L(\mathbf{N}) \subseteq L(\mathbf{U})$ and so this is not a semidirect product of algebraic groups!

For working with fixed points of groups under automorphisms as in 1.1.7, the following completely general result will be useful on several occasions.

Lemma 1.1.9 ([St68, 4.5]) *Let A, B be groups and $f: A \to B$ be a surjective homomorphism with* $\ker(f) \subseteq Z(A)$. *Let $\sigma: A \to A$ and $\tau: B \to B$ be automorphisms such that $f \circ \sigma = \tau \circ f$. Then $C := \{a^{-1}\sigma(a) \mid a \in \ker(f)\}$ is a subgroup of* $\ker(f)$. *Furthermore, let*

$$A^\sigma := \{a \in A \mid \sigma(a) = a\} \qquad \text{and} \qquad B^\tau := \{b \in B \mid \tau(b) = b\}.$$

Then $f(A^\sigma)$ is normal in B^τ and there is a canonical injective homomorphism $\delta: B^\tau/f(A^\sigma) \hookrightarrow \ker(f)/C$, with image $(\{a^{-1}\sigma(a) \mid a \in A\} \cap \ker(f))/C$.

Proof Since $\ker(f) \subseteq Z(A)$ and $f \circ \sigma = \tau \circ f$, it is clear that $\sigma(\ker(f)) \subseteq \ker(f)$ and that C is a subgroup of $\ker(f)$. We define $\delta': B^\tau \to \ker(f)/C$ as follows. Let $b \in B^\tau$ and choose $a \in A$ such that $f(a) = b$. We have $f(\sigma(a)) = \tau(f(a)) = \tau(b) = b$ and so $c := a^{-1}\sigma(a) \in \ker(f)$. Then set $\delta'(b) = cC \in \ker(f)/C$. One verifies that δ' is well defined and a group homomorphism; furthermore, $\ker(\delta') = f(A^\sigma)$ and the image of δ' is as stated above. Thus, we obtain an induced map δ with the required properties. □

1.1.10 The unipotent radical. Let $\mathbf{G}$ be a linear algebraic group over $k = \overline{\mathbb{F}}_p$, where p is a prime. We can now define the *unipotent radical* $R_u(\mathbf{G}) \subseteq \mathbf{G}$, as follows. An abstract subgroup of $\mathbf{G}$ is called *unipotent* if all of its elements are unipotent. Since every element in $\mathbf{G}$ has finite order, one easily sees that the product of two normal unipotent subgroups is again a normal unipotent subgroup of $\mathbf{G}$. If $\mathbf{G}$ is finite, then this immediately shows that there is a unique maximal normal unipotent subgroup in **G**. (In the theory of finite groups, this is denoted $O_p(\mathbf{G})$.) In the general case, we define

$$R_u(\mathbf{G}) := \text{subgroup of } \mathbf{G} \text{ generated by all } \mathbf{U} \in \mathscr{S}_{\text{unip}}(\mathbf{G}),$$

where $\mathscr{S}_{\text{unip}}(\mathbf{G})$ denotes the set of all closed connected normal unipotent subgroups of **G**. It is clear that $R_u(\mathbf{G})$ is an abstract normal subgroup of **G**. By the criterion in 1.1.3, $R_u(\mathbf{G})$ is a closed connected subgroup of **G**; furthermore, $R_u(\mathbf{G}) = \mathbf{U}_1 \dots \mathbf{U}_n$ for some $n \geqslant 1$ and $\mathbf{U}_1, \dots, \mathbf{U}_n \in \mathscr{S}_{\text{unip}}(\mathbf{G})$. As already remarked before, this product will consist of unipotent elements. Thus, $R_u(\mathbf{G})$ is the unique maximal closed connected normal unipotent subgroup of **G**. (The analogous definition also works when k is an arbitrary algebraically closed field, using the slightly more complicated characterisation of unipotent elements in that case.)

We say that **G** *is reductive if* $R_u(\mathbf{G}) = \{1\}$.

(Thus, connected reductive groups can be regarded as analogues of finite groups G with $O_p(G) = \{1\}$.) These are the groups that we will be primarily concerned with. In an arbitrary algebraic group **G**, we always have the closed connected normal subgroups $R_u(\mathbf{G}) \subseteq \mathbf{G}^\circ \subseteq \mathbf{G}$, and $\mathbf{G}/R_u(\mathbf{G})$ will be reductive. Note also that, clearly, we have the implication

$$\mathbf{G} \text{ simple} \quad \Rightarrow \quad \mathbf{G} \text{ reductive (and connected).}$$

Here, $\mathbf{G} \neq \{1\}$ is called a *simple algebraic group* if **G** is connected, non-abelian and if **G** has no closed connected normal subgroups other than $\{1\}$ and **G**. (For example, $\mathrm{SL}_n(k)$ is a simple algebraic group, although in general it is not simple as an abstract group; $\mathrm{GL}_n(k)$ is reductive, but not simple.)

Even if one is mainly interested in studying a simple group **G**, one will also have to look at subgroups with a geometric origin, like Levi subgroups or centralisers of semisimple elements. These subgroups tend to be reductive, not just simple. For example, if **G** is connected and reductive and $s \in \mathbf{G}$ is a semisimple element, then the centraliser $C_{\mathbf{G}}(s)$ will be a closed reductive (not necessarily connected or simple) subgroup; see [Ca85, 3.5.4].

1.1.11 Characters and co-characters of tori. The simplest examples of connected reductive algebraic groups are tori, and it will be essential to understand some basic constructions with them. First, a general definition. A homomorphism of algebraic groups $\lambda\colon \mathbf{G} \to \mathbf{k}^\times$ will be called a *character* of **G**. The set $X = X(\mathbf{G})$ of all characters of **G** is an abelian group (which we write additively), called the *character group* of **G**. Similarly, a homomorphism of algebraic groups $\nu\colon \mathbf{k}^\times \to \mathbf{G}$