

Modular Theory in Operator Algebras

The first edition of this book appeared in 1981 as a direct continuation of *Lectures of von Neumann Algebras* (Ş. V. Strătilă and L. Zsidó) and, until 2003, was the only comprehensive monograph on the subject. Addressing students of mathematics and physics and researchers interested in operator algebras, noncommutative geometry and free probability, this revised edition covers the fundamentals and some latest developments in the field of operator algebras. The intent is to make modular theory accessible, with complete proofs, to readers having elementary training in operator algebras.

The text provides detailed discussion of normal weights, conditional expectations and U. Haagerup's operator-valued weights, groups of automorphisms and their spectral theory, duality theory for noncommutative groups, and crossed products of von Neumann algebras by actions of groups and duals of groups, which enables the extension of M. Takesaki duality for noncommutative groups of automorphisms. It also contains detailed discussion on the group-measure space construction of factors and information about ITPFI factors and Krieger factors.

The core of the book is the continuous decomposition of A. Connes and M. Takesaki and discrete decomposition of A. Connes for type III factors.

It also explores new results, such as the A. Ocneanu's result on the actions of amenable groups on the hyperfinite factor, H. Kosaki's extension of the V. Jones index to arbitrary factors and F. Rădulescu's examples of non-hyperfinite factors of type III_λ , $\lambda \in (0, 1)$ and of type III_1 .

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Modular Theory in Operator Algebras

Second Edition

Șerban Valentin Strătilă



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*To the memory of my wife Sanda Strătilă,
the writer Alexandra Stănescu (1944–2011)*

Contents

<i>Preface to the second edition</i>	<i>ix</i>
<i>Preface to the first edition</i>	<i>xi</i>
Chapter I Normal Weights	1
1 Characterizations of Normality	1
2 The Standard Representation	10
3 The Balanced Weight	33
4 The Pedersen–Takesaki Construction	49
5 The Converse of the Connes Theorem	63
6 Equality and Majorization of Weights	67
7 The Spatial Derivative	77
8 Tensor Products	93
Chapter II Conditional Expectations and Operator-Valued Weights	99
9 Conditional Expectations	99
10 Existence and Uniqueness of Conditional Expectations	111
11 Operator-Valued Weights	129
12 Existence and Uniqueness of Operator-Valued Weights	142
Chapter III Groups of Automorphisms	159
13 Groups of Isometries on Banach Spaces	159
14 Spectra and Spectral Subspaces	165
15 Continuous Actions on W^* -algebras	181
16 The Connes Invariant $\Gamma(\sigma)$	196
17 Outer Automorphisms	208
Chapter IV Crossed Products	229
18 Hopf–von Neumann Algebras	229
19 Crossed Products	261
20 Comparison of cocycles	288
21 Abelian Groups	302
22 Discrete Groups	313

Chapter V Continuous Decompositions	331
23 Dominant Weights and Continuous Decompositions	331
24 The Flow of Weights	348
25 The Fundamental Homomorphism	356
26 The Extension of the Modular Automorphism Group	360
Chapter VI Discrete Decompositions	369
27 The Connes Invariant $T(\mathcal{M})$	369
28 The Connes Invariant $S(\mathcal{M})$	374
29 Factors of Type III_λ ($0 \leq \lambda < 1$)	383
30 The Discrete Decomposition of Factors of Type III_λ ($0 \leq \lambda < 1$)	396
<i>Appendix</i>	415
<i>References</i>	429
<i>Notation Index</i>	443
<i>Subject Index</i>	445

Preface to the Second Edition

The first edition of this book appeared in 1981 as a direct continuation of *Lectures of von Neumann Algebras* by Ş.V. Strătilă and L. Zsidó and, until 2003, was the only comprehensive monograph on the subject.

The book *Lectures on von Neumann Algebras*, 2nd Edition, Cambridge University Press, 2019, will be always referred to as [L]. It is assumed that the reader is familiar with the material contained in this book, including the terminology and notation.

The present book contains the continuous decomposition and the discrete decomposition for factors of type III and all the necessary results such as the extensive theory of normal weights including the U. Haagerup characterization of normality, the A. Connes theorem of Radon–Nikodym type and the Pedersen–Takesaki construction, the conditional expectations and the operator-valued weights, a detailed consideration of groups of automorphisms and their spectral theory, and the theory of crossed products. In order to include our extension of the Takesaki duality theorem to noncommutative groups of automorphisms, we considered a simultaneous generalization of groups and duals of groups, namely the Kac algebras, and their actions on von Neumann algebras, as well as the corresponding crossed products. Instead of Kac algebras we can also consider quantum groups, the arguments being exactly the same as for Kac algebras.

In this second edition we added information and references for several results, which appeared after 1981, until now, including A. Ocneanu’s theorem concerning actions of amenable groups and its extensions, H. Kosaki’s extension of the index to arbitrary factors and F. Rădulescu’s examples of non-hyperfinite factors of type III_λ , $\lambda \in (0, 1)$ and of type III_1 relying on D.-V. Voiculescu’s theory of free probability.

For subjects such as the equivalence of injectivity and hyperfiniteness (originally due to A. Connes) we indicate references for more recent shorter proofs. This guarantees the uniqueness of the hyperfinite factor of type II_∞ and of type III_λ factors, $\lambda \in (0, 1)$. Another important result which could not be considered in detail, due to its length, is the proof of U. Haagerup of the A. Connes conjecture that the hyperfinite factor of type III_1 is unique. For the same reason we restricted ourselves to only a few results concerning Krieger factors and ITPFI factors. However, we considered in detail the “group-measure space construction” which produced many examples of factors, among them the first example of a type III factor by Murray and von Neumann, the Pukanszky non-hyperfinite factor of type III and also the famous Powers factors which are actually the only hyperfinite factors of type III_λ , $\lambda \in (0, 1)$.

I am grateful to Gadadhar Misra and V. Sunder for proposing this second edition, to Gadadhar Misra for his considerable help with the Latex conversion of the book, to Ms Rajitha Reddy for her excellent typing, to Florin Rădulescu for his help for Ocneanu’s theorem and his own theorems, and to Alexandru Negrescu for his help in typing the new material for the second edition.

I am also grateful to Cambridge University Press and to Ms Taranpreet Kaur for editing this second edition.

Şerban Strătilă

Bucharest, January 20, 2020

Preface to the First Edition

The discovery of the modular operator and the modular automorphism group associated with a normal semifinite faithful weight has led to a powerful theory – the modular theory – which is nowadays essential to the consideration of many problems concerning operator algebras. This theory has been developed in close association with the effort to understand the structure and produce examples and refined classifications of factors. Thus, the crossed product construction, which gave rise to the first non-trivial examples of factors, has been shown to play a fundamental role in the structure theory as well, by reducing the study of the purely infinite algebras to the study of the more familiar semifinite algebras and their automorphisms. Moreover, several algebraic invariants, previously defined only in some special cases, have been introduced via the modular theory for arbitrary factors and the corresponding classification has been proved to be almost complete for approximately finite dimensional factors.

The present book is a unified exposition of the technical tools of the modular theory and of its applications to the structure and classification of factors. It is based on several works recently published in periodicals or just circulated as preprints. The main sources used in writing this book are the works of W. B. Arveson, A. Connes, U. Haagerup, M. Landstad, G. K. Pedersen, M. Takesaki, and J. Tomiyama. The general treatment of crossed products follows an article by Ş. Strătilă, D. Voiculescu and L. Zsidó. Due to the wealth and variety of results recently obtained, it has not been possible to include here a detailed exposition of the classification of injective factors and their automorphisms; these topics and several others are just mentioned in the Notes sections, together with appropriate references.

The reader is assumed to have a good knowledge of the general theory of von Neumann algebras, including the standard forms. Actually, the present book can be viewed as a sequel to a previous book, Ş. Strătilă and L. Zsidó – *Lectures on von Neumann algebras*, Editura Academiei & Abacus Press, 1979, which is often quoted here and referred to as [L]. There is also an Appendix which contains some supplementary results on positive self-adjoint operators and introduces the terminology connected with W^* -algebras.

The list of references in the present book contains only those items which have been used, quoted or consulted. A more extensive bibliography is contained in [L] (and in the Preprint Series, INCREST, Bucharest) and the new preprints are periodically recorded in C^* -News (issued by CPT/CNRS, Marseille).

I am very indebted to Zoia Ceaşescu, Alain Connes, Sanda Strătilă and Dan Voiculescu for the moral support they offered me in writing this book. I am grateful to my colleagues Constantin Apostol, Grigore Arsene, Zoia Ceaşescu, Radu Gologan, Adrian Ocneanu, Cornel Pasnicu, Mihai Pimsner, Sorin Popa and Dan Timotin for several useful discussions and a critical reading of various parts of the manuscript. During his short visit in Romania, Alain Connes kindly informed me of the most recent developments of the theory. Thanks are also due to the National Institute for Scientific and Technical Creation, for the technical assistance.

It is a pleasure for me to acknowledge the most efficient and understanding cooperation of the Publishing House of the Romanian Academy (Editura Academiei) and Abacus Press, especially Mrs Sorana Gorjan, who edited this book, and Dr Simon Wassermann of Glasgow University, whose comments on the original translation were most helpful.

Șerban Strătilă

București, Romania, October 1979