CHAPTER I
Normal Weights

1 Characterizations of Normality

In this section, we prove the Theorem of Haagerup asserting that every normal weight on a W^*-algebra is the pointwise least upper bound of the normal positive forms it majorizes.

1.1. Let $\mathscr{A}$ be a C^*-algebra. A *weight* on $\mathscr{A}$ is a mapping $\varphi : \mathscr{A}^+ \to [0, +\infty]$ with the properties

$$\varphi(x + y) = \varphi(x) + \varphi(y), \varphi(\lambda x) = \lambda\varphi(x) \quad (x, y \in \mathscr{A}^+, \lambda \in \mathbb{R}^+).$$

The set

$$\mathfrak{F}_\varphi = \{x \in \mathscr{A}^+; \varphi(x) < +\infty\}$$

is a face of $\mathscr{A}^+$, the set

$$\mathfrak{N}_\varphi = \{x \in \mathscr{A}; \varphi(x^*x) < +\infty\}$$

is a left ideal of $\mathscr{A}$, and the set

$$\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi = \text{lin}\,\mathfrak{F}_\varphi$$

is a facial subalgebra of $\mathscr{A}$ with $\mathfrak{M}_\varphi \cap \mathscr{A}^+ = \mathfrak{F}_\varphi$([L], 3.21), hence φ can be extended uniquely to a positive linear form, still denoted by φ, on the *-algebra $\mathfrak{M}_\varphi$.

A family $\mathscr{F}$ of weights on $\mathscr{A}$ is called *sufficient* if

$$x \in \mathscr{A} \text{ and } \varphi(a^*x^*xa) = 0 \text{ for all } \varphi \in \mathscr{F}, a \in \mathfrak{N}_\varphi \Rightarrow x = 0$$

and is called *separating* if

$$x \in \mathscr{A} \text{ and } \varphi(x^*x) = 0 \text{ for all } \varphi \in \mathscr{F} \Rightarrow x = 0.$$

In particular, the weight φ is called *faithful* if

$$x \in \mathscr{A} \text{ and } \varphi(x^*x) = 0 \Rightarrow x = 0.$$

1.2. Let φ be a weight on the C^*-algebra $\mathscr{A}$. The formula

$$(a|b)_\varphi = \varphi(b^*a) \quad (a, b \in \mathfrak{N}_\varphi)$$

defines a prescalar product on $\mathfrak{N}_\varphi$ with the properties:

$$(xa|xa)_\varphi \le \|x\|^2(a|a)_\varphi \quad (x \in \mathscr{A}, a \in \mathfrak{N}_\varphi),$$
$$(xa|b)_\varphi = (a|x^*b)_\varphi \quad (x \in \mathscr{A}, \quad a, b \in \mathfrak{N}_\varphi).$$

Let $\mathscr{H}_\varphi$ be the Hilbert space associated with $\mathfrak{N}_\varphi$ with the scalar product $(\cdot|\cdot)_\varphi$. It follows that there exists a *-representation $\pi_\varphi : \mathscr{A} \to \mathscr{B}(\mathscr{H}_\varphi)$, uniquely determined, such that

$$(\pi_\varphi(x)a_\varphi|b_\varphi)_\varphi = \varphi(b^*xa) \quad (x \in \mathscr{A}, a, b \in \mathfrak{N}_\varphi), \tag{1}$$

where $\mathfrak{N}_\varphi \ni a \mapsto a_\varphi \in \mathscr{H}_\varphi$ denotes the canonical mapping. The *-representation π_φ is called the *GNS representation* or the *standard representation* associated with φ. We remark that

$$\varphi(x^*) = \overline{\varphi(x)} \quad (x \in \mathfrak{M}_\varphi), \tag{2}$$
$$|\varphi(b^*a)|^2 \le \varphi(a^*a)\varphi(b^*b) \quad (a, b \in \mathfrak{N}_\varphi). \tag{3}$$

1.3. Let $\mathscr{M}$ be a W^*-algebra. A weight φ on $\mathscr{M}$ is called *normal* if

$$\varphi(\sup_i x_i) = \sup_i \varphi(x_i)$$

for every norm-bounded increasing net $\{x_i\}_i \subset \mathscr{M}^+$, and *lower w-semicontinuous* if the convex sets

$$\{x \in \mathscr{M}^+; \varphi(x) \le \lambda\} \quad (\lambda \in \mathbb{R}^+)$$

are w-closed. An important result concerning weights on W^*-algebras is the following characterization of normality:

Theorem (U. Haagerup). *Let φ be a weight on the W^*-algebra $\mathscr{M}$. The following statements are equivalent:*

(i) *φ is normal;*
(ii) *φ is lower w-semicontinuous;*
(iii) $\varphi(x) = \sup\{f(x); f \in \mathscr{M}_*^+, f \le \varphi\}$ *for all* $x \in \mathscr{M}^+$.

Later (2.10, 5.8) we shall see that φ is normal if and only if it is a sum of normal positive forms, in accordance with the definition used in ([L], 10.14).

In Sections 1.4–1.7, we present some general results that will be used in the proof of the theorem; Sections 1.6–1.12 contain the main steps of the proof.

1.4 Proposition. *If $x, x_1, \dots, x_n \in \mathscr{B}(\mathscr{H})$ and $x^*x = x_1^*x_1 + \dots + x_n^*x_n$, then there exist $z_1, \dots, z_n \in \mathscr{R}\{x, x_1, \dots, x_n\}$ such that $z_1^*z_1 + \dots + z_n^*z_n = \mathbf{s}(xx^*)$ and $x_k = z_kx$ for all $1 \le k \le n$.*

Proof. The equations $z_k(x\xi) = x_k\xi(\xi \in \mathscr{H})$ and $z_k\eta = 0(\eta \in \mathscr{H} \bigoplus \overline{x\mathscr{H}})$ define operators $z_k \in \mathscr{B}(\mathscr{H})$, $\|z_k\| \le 1$, with $x_k = z_kx$ and $z_k(\mathscr{H} \ominus \overline{x\mathscr{H}}) = 0$. Using the double commutant theorem

([L], 3.2) it is easy to check that $z_k \in \mathscr{R}\{x, x_k\}$. Also the relation $\sum_k z_k^* z_k = \mathbf{s}(xx^*)$ follows, since the positive operator $(\sum_k z_k^* z_k)^{1/2}$ vanishes on $\mathscr{H} \ominus \overline{x\mathscr{H}}$ and $x^*(\sum_k z_k^* z_k)x = x^*x$.

In particular if $x, y \in \mathscr{B}(\mathscr{H})$ and $y^*y \leq x^*x$, there exists $z \in \mathscr{R}\{x, y\}$ such that $z^*z \leq \mathbf{s}(xx^*)$ and $y = zx$. □

1.5. For each $\alpha > 0$ we shall consider the function

$$f_\alpha : (-\alpha^{-1}, +\infty) \to \mathbb{R}$$

defined by $f_\alpha(t) = t(1+\alpha t)^{-1} = a^{-1}(1-(1+\alpha t)^{-1})$. These functions have the following properties:

$$f_\alpha(t) \leq \min\{t, \alpha^{-1}\} \qquad (t \in (-\alpha^{-1}, +\infty)) \tag{1}$$
$$\alpha \leq \beta \Rightarrow f_\alpha(t) \geq f_\beta(t) \qquad (t \in (-\alpha^{-1}, +\infty)) \tag{2}$$
$$\alpha \leq \beta \Rightarrow \alpha f_\alpha(t) \leq \beta f_\beta(t) \qquad (t \in (0, +\infty)) \tag{3}$$
$$f_\alpha(f_\beta(t)) = f_{\alpha+\beta}(t) \qquad (t \in (-(\alpha+\beta)^{-1}, +\infty)) \tag{4}$$
$$\lim_{\alpha \to 0} f_\alpha(t) = t \textit{ uniformly on compact subsets of } \mathbb{R} \tag{5}$$
$$\lim_{\alpha \to \infty} \alpha f_\alpha(t) = 1 \textit{ uniformly on compact subsets of } \mathbb{R}^+. \tag{6}$$

A continuous function $f : I \to \mathbb{R}$ is called *operator monotone* on the interval $I \subset \mathbb{R}$ if for every $x, y \in \mathscr{B}(\mathscr{H}), x = x^*, y = y^*$, with $Sp(x) \subset I, Sp(y) \subset I$, we have $x \leq y \Rightarrow f(x) \leq f(y)$. For instance, it is easy to see that

$$\text{the functions } f_\alpha \text{ are operator monotone } (\alpha \in \mathbb{R}^+). \tag{7}$$

Also, we recall that (see Pedersen, 1973–1977 or Strătilă & Zsidó, 1977, 1979)

$$\text{the functions } t \to t^\gamma \text{ are operator monotone } (0 < \gamma < 1). \tag{8}$$

On the other hand, using (A.2) we see that

$$\text{the functions } f_\alpha \text{ are operator continuous } (\alpha \in \mathbb{R}^+). \tag{9}$$

1.6. Let $\mathscr{X}$ be a locally convex Hausdorff real vector space with a partial ordering defined by a convex cone $\mathscr{X}^+ \subset \mathscr{X}$ such that $\mathscr{X}^+ \cap (-\mathscr{X}^+) = \{0\}$ and $\mathscr{X} = (\overline{\mathscr{X}^+ - \mathscr{X}^+})$. The dual cone $\mathscr{X}_+^* = \{f \in \mathscr{X}^*; f(x) \geq 0 \text{ for all } x \in \mathscr{X}^+\}$ defines a partial ordering on $\mathscr{X}^*$. A subset $\mathscr{E}$ of $\mathscr{X}^+$ is called *hereditary* if

$$x \in \mathscr{E}, y \in \mathscr{X}^+, x - y \in \mathscr{X}^+ \Rightarrow y \in \mathscr{E}.$$

For $\mathscr{E} \subset \mathscr{X}^+$ and $\mathscr{F} \subset \mathscr{X}_+^*$, we define $\mathscr{E}^\wedge$ and $\mathscr{F}^\wedge$ by

$$\mathscr{E}^\wedge = \{f \in \mathscr{X}_+^*; f(x) \leq 1 \text{ for all } x \in \mathscr{E}\},$$
$$\mathscr{F}^\wedge = \{x \in \mathscr{X}^+; f(x) \leq 1 \text{ for all } f \in \mathscr{F}\}.$$

Proposition. *For $\mathscr{X}$ as above the following statements are equivalent:*

(i) $\mathscr{E} = (\overline{\mathscr{E} - \mathscr{X}^+}) \cap \mathscr{X}^+$ *for every closed hereditary convex subset* $\mathscr{E}$ *of* $\mathscr{X}^+$;
(ii) $\mathscr{E} = \mathscr{E}^{\wedge\wedge}$ *for every closed hereditary convex subset* $\mathscr{E}$ *of* $\mathscr{X}^+$;
(iii) *every subadditive, positively homogeneous, increasing and lower semicontinuous function* $\varphi : \mathscr{X}^+ \to [0, +\infty]$ *has the property*

$$\varphi(x) = \sup\{f(x); f \in \mathscr{X}_+^*, f \leq \varphi\} \quad (x \in \mathscr{X}^+).$$

Proof. We shall denote by $\mathscr{S}^0$ the polar of a subset $\mathscr{S}$ of $\mathscr{X}$ or $\mathscr{X}^*$.

(i) $\Rightarrow$ (ii). The sets $\mathscr{F} = \mathscr{E}^\wedge$ and $\mathscr{F}' = -(\mathscr{E} - \mathscr{X}^+)^0 = \{f \in \mathscr{X}^*; f(x) \leq 1$ for all $x \in \mathscr{E} - \mathscr{X}^+\}$ are equal. Indeed, it is clear that $\mathscr{F} \subset \mathscr{F}'$. Let $f \in \mathscr{F}'$ and $x \in \mathscr{X}^+$. Since $0 \in \mathscr{E}$, we have $f(-\lambda x) \leq 1$ for all $\lambda \geq 0$, whence $f(x) \geq 0$. Thus $\mathscr{F}' \subset \mathscr{X}_+^*$ and so $\mathscr{F}' \subset \mathscr{F}$.

By the bipolar theorem it follows that $(\overline{\mathscr{E} - \mathscr{X}^+}) = (\mathscr{E} - \mathscr{X}^+)^{00} = (-\mathscr{F})^0 = \{x \in \mathscr{X}; f(x) \leq 1$ for all $f \in \mathscr{F}\}$ and, using (i), we get $\mathscr{E} = (\overline{\mathscr{E} - \mathscr{X}^+}) \cap \mathscr{X}^+ = \{x \in \mathscr{X}^+; f(x) \leq 1$ for all $f \in \mathscr{F}\} = \mathscr{E}^{\wedge\wedge}$.

(ii) $\Rightarrow$ (iii). If φ satisfies the conditions required in (iii), then the set $\mathscr{E} = \{x \in \mathscr{X}^+; \varphi(x) \leq 1\}$ is closed, hereditary and convex. Also, $\mathscr{F} = \mathscr{E}^\wedge = \{f \in \mathscr{X}_+^*; f(x) \leq \varphi(x)$ for all $x \in \mathscr{X}^+\}$ and, by (ii), $\{x \in \mathscr{X}^+; \varphi(x) \leq 1\} = \mathscr{E} = \mathscr{F}^\wedge = \{x \in \mathscr{X}^+; \sup_{f \in \mathscr{F}} f(x) \leq 1\}$. It follows that $\varphi(x) = \sup\{f(x); f \in \mathscr{F}\}$, for all $x \in \mathscr{X}^+$.

(iii) $\Rightarrow$ (i). Let $\mathscr{E} \subset \mathscr{X}^+$ be closed, hereditary and convex. Define $\varphi(x) = \inf\{\lambda > 0; x \in \lambda\mathscr{E}\}$ if $x \in \bigcup_{\lambda > 0} \lambda\mathscr{E}$ and $\varphi(x) = +\infty$ otherwise. Then φ satisfies the hypotheses in (iii) and therefore $\varphi(x) = \sup\{f(x); f \in \mathscr{F}\} (x \in \mathscr{X}^+)$, where $\mathscr{F} = \{f \in \mathscr{X}_+^*; f(x) \leq \varphi(x)$ for all $x \in \mathscr{X}^+\}$. It follows that $\mathscr{E} - \mathscr{X}^+ \subset \{x \in \mathscr{X}; f(x) \leq 1$ for all $f \in \mathscr{F}\}$ and, since the latter set is closed, we get $(\overline{\mathscr{E} - \mathscr{X}^+}) \cap \mathscr{X}^+ \subset \{x \in \mathscr{X}^+; f(x) \leq 1$ for all $f \in \mathscr{F}\} \subset \mathscr{E}$, hence $(\overline{\mathscr{E} - \mathscr{X}^+}) \cap \mathscr{X}^+ = \mathscr{E}$. □

1.7 Proposition. *Let $\mathscr{M}$ be a W^*-algebra and $\mathscr{E} \subset \mathscr{M}^+$ a w-closed hereditary convex set. Then $\mathscr{E} = (\overline{\mathscr{E} - \mathscr{M}^+})^w \cap \mathscr{M}^+$.*

Proof. We shall use the properties of the functions f_α from 1.5. For $x \in \mathscr{M}_h$ let $\alpha_x = \sup\{\alpha > 0; -\alpha^{-1} \leq x\}$. Consider the set

$$\mathscr{S} = \{x \in \mathscr{M}_h; f_\alpha(x) \in \mathscr{E} - \mathscr{M}^+ \text{ for all } \alpha \in (0, \alpha_x)\},$$

and let $\mathscr{M}_\lambda; . = \{x \in \mathscr{M}; \|x\| \leq \lambda\}$.

We first show that for every $\lambda > 0$ the set $\mathscr{S} \cap \mathscr{M}_\lambda$ is s-closed.

Indeed, let $x \in \overline{\mathscr{S} \cap \mathscr{M}_\lambda}^s$. There is a net $\{x_i\}_{i \in I} \subset \mathscr{S}$ such that $\|x_i\| \leq \lambda$ and $x_i \xrightarrow{s} x$. Then $\alpha_{x_i} \geq 1/\lambda$, hence $f_\alpha(x_i) \in \mathscr{E} - \mathscr{M}^+$ for every $\alpha \in (0, 1/2\lambda)$ and every $i \in I$. Let $\alpha \in (0, 1/2\lambda)$ be fixed. There is a net $\{y_i\}_{i \in I} \subset \mathscr{E}$ such that

$$f_\alpha(x_i) \leq y_i \quad (i \in I).$$

Since f_α is operator monotone,

$$f_{2\alpha}(x_i) = f_\alpha(f_\alpha(x_i)) \leq f_\alpha(y_i) \quad (i \in I).$$

Since $f_{2\alpha}$ is operator continuous on $[-\lambda, +\lambda]$,

$$f_{2\alpha}(x_i) \xrightarrow{s} f_{2\alpha}(x).$$

Since $0 \leq f_\alpha(y_i) \leq \alpha^{-1}$ and $\mathscr{M}_1$ is w-compact, we may assume that there is $y \in \mathscr{M}$ such that

$$f_\alpha(y_i) \xrightarrow{w} y.$$

Since $0 \leq f_\alpha(y_i) \leq y_i \in \mathscr{E}$ and $\mathscr{E}$ is hereditary, $f_\alpha(y_i) \in \mathscr{E}$ and, since $\mathscr{E}$ is w-closed, it follows that $y \in \mathscr{E}$. Then

$$y - f_{2\alpha}(x) = w\text{-}\lim_i (f_\alpha(y_i) - f_{2\alpha}(x_i)) \geq 0,$$

hence $f_{2\alpha}(x) \in \mathscr{E} - \mathscr{M}^+$. We have thus proved that

$$f_\alpha(x) \in \mathscr{E} - \mathscr{M}^+ \text{ for every } \alpha \in (0, 1/\lambda).$$

Consider now $\alpha \in [1/\lambda, \alpha_x)$ and $\beta \in (0, 1/\lambda)$. Then $f_\alpha(x) \leq f_\beta(x)$, hence $f_\alpha(x) \in (\mathscr{E} - \mathscr{M}^+) - \mathscr{M}^+ = \mathscr{E} - \mathscr{M}^+$. We conclude that $x \in \mathscr{S} \cap \mathscr{M}_\lambda$. □

We now show that $\mathscr{S}$ is convex.

Indeed, it is sufficient to show that each $\mathscr{S} \cap \mathscr{M}_\lambda$ is convex, and this will follow from the equality

$$\mathscr{S} \cap \mathscr{M}_\lambda = \overline{((\mathscr{E} - \mathscr{M}^+) \cap \mathscr{M}_\mu)}^s \cap \mathscr{M}_\lambda \quad \text{for } \mu > \lambda.$$

If $x \in \mathscr{S} \cap \mathscr{M}_\lambda$, then $f_\alpha(x) \in \mathscr{E} - \mathscr{M}^+$ for $\alpha \in (0, \alpha_x)$ and $f_\alpha(x) \in \mathscr{M}_\mu$ for small $\alpha > 0$, hence

$$x = s\text{-}\lim_{\alpha \to 0} f_\alpha(x) \in \overline{((\mathscr{E} - \mathscr{M}^+) \cap \mathscr{M}_\mu)}^s \cap \mathscr{M}_\lambda.$$

Conversely, since $\mathscr{E}$ is hereditary and $f_\alpha(x) \leq x$ for all $\alpha \in (0, \alpha_x)$, we have $\mathscr{E} - \mathscr{M}^+ \subset \mathscr{S}$, hence $(\mathscr{E} - \mathscr{M}^+) \cap \mathscr{M}_\mu \subset \mathscr{S} \cap \mathscr{M}_\mu$. Using the first part of the proof we get $\overline{((\mathscr{E} - \mathscr{M}^+) \cap \mathscr{M}_\mu)}^s \subset \mathscr{S} \cap \mathscr{M}_\mu$, and the desired inclusion follows.

Using the Krein-Šmulian theorem ([L], C.1.1; Dunford & Schwartz, 1958, 1963, V.5.7), from the earlier it follows that $\mathscr{S}$ is w-closed. We have seen that $\mathscr{E} - \mathscr{M}^+ \subset \mathscr{S}$. Since $x = w\text{-}\lim_{\alpha \to 0} f_\alpha(x)$, we obtain $\mathscr{S} \subset \overline{(\mathscr{E} - \mathscr{M}^+)}^w$. Consequently, $\mathscr{S} = \overline{(\mathscr{E} - \mathscr{M}^+)}^w$.

Finally, let $x \in \overline{(\mathscr{E} - \mathscr{M}^+)}^w \cap \mathscr{M}^+ = \mathscr{S} \cap \mathscr{M}^+$. For every $\alpha > 0$ we have $f_\alpha(x) \in (\mathscr{E} - \mathscr{M}^+) \cap \mathscr{M}^+$, hence $f_\alpha(x) \in \mathscr{E}$, as $\mathscr{E}$ is hereditary. It follows that $x = w\text{-}\lim_{\alpha \to 0} f_\alpha(x) \in \mathscr{E}$.

From Propositions 1.6 and 1.7, we obtain the equivalence (ii) ⇒ (iii) in Theorem 1.3, as the implication (iii) ⇒ (ii) is obvious.

In Sections 1.8–1.12, we assume that φ is a fixed normal weight on the W^*-algebra $\mathscr{M}$.

1.8 Lemma. *There exists a linear mapping* $\Phi : \mathfrak{M}_\varphi \to \pi_\varphi(\mathscr{M})'_*$, *uniquely determined, such that*

$$\Phi(b^* a)(T') = (T' a_\varphi | b_\varphi)_\varphi \quad (T' \in \pi_\varphi(\mathscr{M})', a, b \in \mathfrak{N}_\varphi). \tag{1}$$

Moreover, for every $x \in \mathfrak{M}_\varphi \cap \mathscr{M}_h$ *we have*

$$\|\Phi(x)\| = \inf\{\varphi(y) + \varphi(z); y, z \in \mathfrak{M}_\varphi \cap \mathscr{M}^+, x = y - z\}. \tag{2}$$

Proof. The uniqueness of Φ follows from the relation $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^* \mathfrak{N}_\varphi$.

If $a, b, c \in \mathfrak{N}_\varphi, c^* = c$ and $c^*c = a^*a + b^*b$, then, by Proposition 1.4, there exist $x, y \in \mathscr{M}$ such that $a = xc, b = yc$ and $x^*x + y^*y = \mathbf{s}(cc^*) = \mathbf{s}(c)$, and for every $T' \in \pi_\varphi(\mathscr{M})'$ we have

$$\begin{aligned}(T'c_\varphi|c_\varphi)_\varphi &= (T'\pi_\varphi(x^*x + y^*y)c_\varphi|c_\varphi)_\varphi \\ &= (T'\pi_\varphi(x)c_\varphi|\pi_\varphi(x)c_\varphi)_\varphi + (T'\pi_\varphi(y)c_\varphi|\pi_\varphi(y)c_\varphi)_\varphi \\ &= (T'a_\varphi|a_\varphi)_\varphi + (T'b_\varphi|b_\varphi)_\varphi.\end{aligned}$$

It follows that the mapping

$$\Phi_0 : \mathfrak{M}_\varphi \cap \mathscr{M}^+ \ni a^*a \mapsto \omega_{a_\varphi}|\pi_\varphi(\mathscr{M})' \in \pi_\varphi(\mathscr{M})'_*.$$

is well defined and additive. Clearly, Φ_0 is positively homogeneous. Since $\mathfrak{M}_\varphi = \mathrm{lin}(\mathfrak{M}_\varphi \cap \mathscr{M}^+)$, Φ_0 has a unique linear extension Φ to $\mathfrak{M}_\varphi$ and (1) follows using the polarization relation ([L], p. 75).

The function ρ defined on $\mathfrak{M}_\varphi \cap \mathscr{M}_h$ by the right-hand side of (2) is a semi-norm on $\mathfrak{M}_\varphi \cap \mathscr{M}_h$. If $x \in \mathfrak{M}_\varphi \cap \mathscr{M}^+$, then $\|\Phi(x)\| = \Phi(x)(1) = ((x^{1/2})_\varphi|(x^{1/2})_\varphi)_\varphi = \varphi(x) = \rho(x)$. Consequently, for $x = y - z$, with $y, z \in \mathfrak{M}_\varphi \cap \mathscr{M}^+$, we have $\|\Phi(x)\| \le \|\Phi(y)\| + \|\Phi(z)\| = \varphi(y) + \varphi(z)$. Hence $\|\Phi(x)\| \le \rho(x)$ for all $x \in \mathfrak{M}_\varphi \cap \mathscr{M}_h$.

Let $x_0 \in \mathfrak{M}_\varphi \cap \mathscr{M}_h$. By the Hahn–Banach theorem, there exists a real linear form f on $\mathfrak{M}_\varphi \cap \mathscr{M}_h$ such that $f(x_0) = \rho(x_0)$ and $|f(x)| \le \rho(x)$ for every $x \in \mathfrak{M}_\varphi \cap \mathscr{M}_h$. Then f can be extended to a complex linear form, still denoted by f, on $\mathfrak{M}_\varphi$. Since $-\varphi(x) \le f(x) \le \varphi(x)$ for any $x \in \mathfrak{M}_\varphi \cap \mathscr{M}^+$, we may consider $\varphi + f$ and $\varphi - f$ as weights on $\mathscr{M}$. Consequently, using the Schwarz inequality 1.2.(3), for $a, b \in \mathfrak{N}_\varphi$ we obtain

$$\begin{aligned}f(b^*a) &\le 2^{-1}[|(\varphi + f)(b^*a)| + |(\varphi - f)(b^*a)|] \\ &\le 2^{-1}[(\varphi + f)(a^*a)^{1/2}(\varphi + f)(b^*b)^{1/2} + (\varphi - f)(a^*a)^{1/2}(\varphi - f)(b^*b)^{1/2}] \\ &\le 2^{-1}[(\varphi + f)(a^*a) + (\varphi - f)(a^*a)]^{1/2}[(\varphi + f)(b^*b) + (\varphi - f)(b^*b)]^{1/2} \\ &= \varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2} = \|a_\varphi\|_\varphi\|b_\varphi\|_\varphi.\end{aligned}$$

Thus, there exists an operator $T' \in \mathscr{B}(\mathscr{H}_\varphi), \|T'\| \le 1$, such that $f(b^*a) = (T'a_\varphi/b_\varphi)_\varphi$ for all $a, b \in \mathfrak{N}_\varphi$. Moreover $T' \in \pi_\varphi(\mathscr{M})'$, since for every $x \in \mathscr{M}$ and every $a, b \in \mathfrak{N}_\varphi$ we have

$$(T'\pi_\varphi(x)a_\varphi|b_\varphi)_\varphi = f(b^*xa) = (\pi_\varphi(x)T'a_\varphi|b_\varphi)_\varphi.$$

It follows that $\rho(x_0) = |f(x_0)| = |\Phi(x_0)(T')| \le \|\Phi(x_0)\|\|T'\| \le \|\Phi(x_0)\|$. □

1.9 Lemma. *Let* $\{x_n\}$ *be a norm-bounded sequence in* $\mathfrak{M}_\varphi \cap \mathscr{M}^+$ *such that the sequence* $\{\Phi(x_n)\}$ *is norm-convergent in* $\pi_\varphi(\mathscr{M})'_*$*. Then*

$$x_n \xrightarrow{s} x \in \mathscr{M} \Rightarrow x \in \mathfrak{M}_\varphi \cap \mathscr{M}^+, \tag{1}$$

$$x_n \xrightarrow{s} 0 \Rightarrow \|\Phi(x_n)\| \to 0. \tag{2}$$

Proof. Let $\varepsilon > 0$ and $\psi = \lim_n \Phi(x_n) \in \pi_\varphi(\mathscr{M})'_*$. Without loss of generality we may assume that $\|\Phi(x_n) - \psi\| < \varepsilon/2^n$, so that $\|\Phi(x_{n+1} - x_n)\| < \varepsilon/2^{n-1}$ for all $n \in \mathbb{N}$. By Lemma 1.8, there exist sequences $\{y_n\}$ and $\{z_n\}$ in $\mathfrak{M}_\varphi \cap \mathscr{M}^+$ such that

$$x_{n+1} - x_n = y_n - z_n \text{ and } \varphi(y_n) + \varphi(z_n) < \varepsilon/2^{n-1} \quad (n \in \mathbb{N}).$$

We shall again use the functions f_α from Section 1.5.

(1) Since $x_{n+1} \le x_1 + \sum_{k=1}^n y_k$ and $x_{n+1} \xrightarrow{s} x$, we obtain

$$f_\alpha(x) = s\text{-}\lim_n f_\alpha(x_{n+1}) \le \sup_n f_\alpha\left(x_1 + \sum_{k=1}^n y_k\right)$$

and then, using the normality of φ,

$$\begin{aligned}\varphi(f_\alpha(x)) &\le \sup_n \varphi\left(f_\alpha\left(x_1 + \sum_{k=1}^n y_k\right)\right) \le \sup_n \varphi\left(x_1 + \sum_{k=1}^n y_k\right)\\ &\le \varphi(x_1) + \sum_{k=1}^\infty \varphi(y_k) \le \varphi(x_1) + \sum_{k=1}^\infty \varepsilon/2^{k-1} = \varphi(x_1) + 2\varepsilon.\end{aligned}$$

Since $f_\alpha(x) \uparrow x$, again using the normality of φ we get

$$\varphi(x) = \sup_{\alpha>0} \varphi(f_\alpha(x)) \le \varphi(x_1) + 2\varepsilon < +\infty,$$

hence $x \in \mathfrak{M}_\varphi \cap \mathscr{M}^+$.

(2) Since $-\sup_n \|x_n\| \le x_1 - x_{n+1} \le \sum_{k=1}^n z_k$, for $\alpha > (\sup_n \|x_n\|)^{-1}$ we obtain

$$f_\alpha(x_1 - x_{n+1}) \le \sup_n f_n\left(\sum_{k=1}^n z_k\right).$$

Since $x_1 - x_{n+1} \xrightarrow{s} x_1$, it follows that

$$f_\alpha(x_1) = s\text{-}\lim_n f_\alpha(x_1 - x_{n+1}) \le \sup_n f_\alpha\left(\sum_{k=1}^n z_k\right).$$

Using the normality of φ we infer that

$$\begin{aligned}\varphi(x_1) &= \sup_{\alpha>0} \varphi(f_\alpha(x_1)) \le \sup_{\alpha>0} \sup_n \varphi\left(f_\alpha\left(\sum_{k=1}^n z_k\right)\right)\\ &\le \sup_n \varphi\left(\sum_{k=1}^n z_k\right) \le \sum_{k=1}^\infty \varepsilon/2^{k-1} = 2\varepsilon.\end{aligned}$$

Consequently, $\|\psi\| \le \|\psi - \Phi(x_1)\| + \|\Phi(x_1)\| \le \varepsilon/2 + 2\varepsilon = 3\varepsilon/2$. We conclude that $\psi = 0$. □

1.10. Let $\mathscr{G}_\varphi = \{(x, x_\varphi); x \in \mathfrak{N}_\varphi\} \subset \mathscr{M} \times \mathscr{H}_\varphi$. Since every Hilbert space is a reflexive Banach space, $\mathscr{M} \times \mathscr{H}_\varphi$ is the dual of the Banach space $\mathscr{M}_* \times \mathscr{H}_\varphi$. For $\lambda, \mu > 0$, let $\mathscr{M}_\lambda = \{x \in \mathscr{M}; \|x\| \le \lambda\}$ and $(\mathscr{H}_\varphi)_\mu = \{\xi \in \mathscr{H}_\varphi; \|\xi\| \le \mu\}$.

Lemma. *If $\mathscr{M}$ is countably decomposable, then $\mathscr{G}_\varphi \cap (\mathscr{M}_\lambda \times (\mathscr{H}_\varphi)_\mu)$ is $\sigma(\mathscr{M} \times \mathscr{H}_\varphi, \mathscr{M}_* \times \mathscr{H}_\varphi)$-compact, for ever $\mu > 0$.*

Proof. Since $\mathscr{G}_\varphi \cap (\mathscr{M}_\lambda \times (\mathscr{H}_\varphi)_\mu)$ is convex and bounded, it is sufficient to prove that it is closed with respect to the product topology τ on $\mathscr{M} \times (\mathscr{H}_\varphi)$ of the s^*-topology on $\mathscr{M}$ and the norm-topology on $\mathscr{H}_\varphi$. Since $\mathscr{M}$ is countably decomposable, $\mathscr{M}_\lambda$ is s^*-metrizable ([L], E.5.7; Strătilă & Zsidó, 1977, 1979, 8.12). □

If $(x, \xi) \in \mathscr{M} \times \mathscr{H}_\varphi$ is τ-adherent to $\mathscr{G}_\varphi \cap (\mathscr{M}_\lambda \times (\mathscr{H}_\varphi)_\mu)$, then there exists a sequence $\{x_n\}$ in $\mathscr{M}_\lambda$ such that $x_n \xrightarrow{s^*} x$, $\|(x_n)_\varphi\|_\varphi \le \mu$ and $\|(x_n)_\varphi - \xi\| \to 0$. Then $x_n^* x_n \xrightarrow{s^*} x^* x$ and $\Phi(x_n^* x_n) = \omega_{(x_n)_\varphi}$ is norm convergent to ω_ξ, whence $x \in \mathfrak{N}_\varphi$ by Lemma 1.8.(1). On the other hand, $(x_n - x)^*(x_n - x) \xrightarrow{s} 0$ and

$$\Phi((x_n - x)^*(x_n - x)) = \omega_{(x_n)_\varphi - x_\varphi} \to \omega_{\xi - x_\varphi}$$

so that $\omega_{\xi - x_\varphi} = 0$ by Lemma 1.8.(2). Thus $\xi = x_\varphi$ and $(x, \xi) \in \mathscr{G}_\varphi$.

1.11. If $\mathscr{M}$ is not countably decomposable, we consider the set $\mathscr{P}_0$ of all countably decomposable projections of $\mathscr{M}$ and put

$$\mathscr{M}_0 = \bigcup_{p \in \mathscr{P}_0} p\mathscr{M}p.$$

It is easy to check that $\mathscr{M}_0$ is a self-adjoint ideal in $\mathscr{M}$.

Lemma. *Let $\mathscr{E}$ be a hereditary convex subset of $\mathscr{M}_0 \cap \mathscr{M}^+$. Then $\mathscr{E}$ is w-closed in $\mathscr{M}_0$ if and only if $\mathscr{E} \cap p\mathscr{M}p$ is w-closed for every $p \in \mathscr{P}_0$.*

Proof. Assume that $\mathscr{E} \cap p\mathscr{M}p$ is w-closed for every $p \in \mathscr{P}_0$. The set $\mathscr{F} = \{x \in \mathscr{M}; x^*x \in \mathscr{E}\}$ is convex and $a\mathscr{F} \subset \mathscr{F}$ for all $a \in \mathscr{M}_1$.

We first show that $p\mathscr{F}$, or equivalently $\mathscr{F}^*p$, is w-closed for any $p \in \mathscr{P}_0$. Using the Krein-Šmulian theorem and the fact that any s-closed convex set is also w-closed, it is sufficient to show that $\mathscr{F}^*p \cap \mathscr{M}_\lambda$ is s-closed for every $\lambda > 0$. Let $x \in \mathscr{M}$ be such that x^* is s-adherent to $\mathscr{F}^*p \cap \mathscr{M}_\lambda$. Since $\mathscr{M}p \cap \mathscr{M}_\lambda$ is s-metrizable, there exists a sequence $\{x_n\}$ in $p\mathscr{F}$, $\|x_n\| \le \lambda$, with $x_n^* \xrightarrow{s} x^*$. There exists a projection $q \in \mathscr{P}_0$ such that $x_n \in q\mathscr{M}q$ for all $n \in \mathbb{N}$. Thus

$$x_n \in \mathscr{F} \cap q\mathscr{M}q = \{y \in q\mathscr{M}q; y^*y \in \mathscr{E} \cap q\mathscr{M}q\} \quad (n \in \mathbb{N}).$$

By assumption, $\mathscr{E} \cap q\mathscr{M}q$ is w-closed, hence $\mathscr{F} \cap q\mathscr{M}q$ is s-closed. It follows that $\mathscr{F} \cap q\mathscr{M}q$ is w-closed and, consequently, $x \in \mathscr{F} \cap q\mathscr{M}q$. Since $px = x$ and $\|x\| \le \lambda$, we get $x^* \in \mathscr{F}^*p \cap \mathscr{M}_\lambda$. Hence $p\mathscr{F}$ is w-closed. □

Let $x \in \mathscr{M}_0$ be w-adherent to $\mathscr{E}$. There exists a net $\{x_i\}_{i\in I} \subset \mathscr{E}$ such that $x_i \xrightarrow{s} x$. Then $p = \mathbf{l}(x) \in \mathscr{P}_0$ and $px_i^{1/2} \xrightarrow{s} px^{1/2} = x^{1/2}$. By the earlier paragraph, we know that $p\mathscr{F}$ is s-closed, hence $x^{1/2} \in p\mathscr{F} \subset \mathscr{F}$, that is, $x \in \mathscr{E}$. Hence $\mathscr{E}$ is w-closed in $\mathscr{M}_0$.

The converse is obvious.

1.12. *Proof of Theorem* 1.3. As we have already seen (1.7), (ii) $\Leftrightarrow$ (iii). The implication (ii) $\Rightarrow$ (i) is obvious. To show that (i) $\Rightarrow$ (ii), we have to prove that the set

$$\mathscr{E} = \{x \in \mathscr{M}^+; \varphi(x) \le 1\}$$

is w-closed. Clearly, $\mathscr{E}$ is hereditary and convex.

Assume first that $\mathscr{M}$ is countably decomposable. As in the last part of the proof of Lemma 1.11, it is sufficient to show that the set $\mathscr{F} = \{x \in \mathscr{M}; \varphi(x^*x) \le 1\}$ is w-closed. Since $\mathscr{F} \cap \mathscr{M}_\lambda$ is the image of $\mathscr{G}_\varphi \cap (\mathscr{M}_\lambda \times (\mathscr{H}_\varphi)_1)$ by the canonical projection mapping $(x, \xi) \mapsto x$, from Lemma 1.10 it follows that $\mathscr{F} \cap \mathscr{M}_\lambda$ is w-compact for every $\lambda > 0$. Since $\mathscr{F}$ is convex, we infer that $\mathscr{F}$ is w-closed.

Consider now the general case. By the earlier argument and by Lemma 1.11, it follows that $\mathscr{E} \cap \mathscr{M}_0$ is w-closed in $\mathscr{M}_0$. Let $x \in \mathscr{M}^+$ be w-adherent to $\mathscr{E}$. There exists a net $\{x_i\}_{i\in I} \subset \mathscr{E}$ such that $x_1 \xrightarrow{s} x$. Also, there exists an increasing net $\{p_k\}_{k\in K} \subset \mathscr{P}_0$ with $p_k \uparrow 1$. Since $\mathscr{M}_0$ is a two-sided ideal in $\mathscr{M}$, for every $k \in K$ we have

$$\mathscr{E} \cap \mathscr{M}_0 \ni x_i^{1/2} p_k x_i^{1/2} \xrightarrow{w}_{i\in I} x^{1/2} p_k x^{1/2} \in \mathscr{M}_0,$$

hence $x^{1/2}p_kx^{1/2} \in \mathscr{E} \cap \mathscr{M}_0$. Since $x^{1/2}p_kx^{1/2} \uparrow x$, using the normality of φ we infer that $\varphi(x) = \sup_{k\in K} \varphi(x^{1/2}p_kx^{1/2}) \le 1$, that is, $x \in \mathscr{E}$.

1.13. We recall that a positive form φ on the W^*-algebra $\mathscr{M}$ is normal if and only if it is completely additive on projections ([L], 5.6, 5.11). This statement cannot be extended to weights, as the following example shows.

Let $\ell^\infty(\mathbb{N})$ be the W^*-algebra of all bounded complex sequences. The weight φ defined on $\ell^\infty(\mathbb{N})$ by $\varphi(\{a_n\}) = \sum_n a_n$ if the set $\{n \in \mathbb{N}; a_n \ne 0\}$ is finite, and $\varphi(\{a_n\}) = +\infty$ otherwise, is completely additive on projections, but is not normal.

1.14 Proposition. *Let φ be a normal weight on the W^*-algebra $\mathscr{M}$ and $a, b \in \mathfrak{N}_\varphi$. Then the mapping*

$$\varphi(b^* \cdot a) : \mathscr{M} \ni x \mapsto \varphi(b^*xa) \in \mathbb{C}$$

is a w-continuous linear form on $\mathscr{M}$.

Proof. Since $a, b \in \mathfrak{N}_\varphi$, for any $x \in \mathscr{M}$ we have $b^*xa \in \mathscr{R}_\varphi^*\mathfrak{N}_\varphi = \mathfrak{M}_\varphi$, hence $\varphi(b^* \cdot a)$ is well defined. If $x_i \uparrow x$ in $\mathscr{M}^+$, then $a^*x_ia \uparrow a^*xa$ in $\mathscr{M}^+$, and hence $\varphi(a^*x_ia) \uparrow \varphi(a^*xa)$, since φ is normal. It follows that $\varphi(a^* \cdot a)$ is w-continuous ([L], 5.11) and the general case is obtained using a polarization relation ([L], 3.21). □

1.15. Notes. The main result (Thm. 1.3) of this section is due to Haagerup (1975a).

For our exposition we have used Haagerup (1975a) and [L].

2 The Standard Representation

In this section, we prove that every normal semifinite weight is the supremum of an upward directed family of normal positive forms; also, we review and complete the results in ([L], Chapter 10) concerning the associated standard representation.

2.1. Let φ be a normal weight on the W^*-algebra $\mathscr{M}$.

Using ([L], 2.22) and the normality of φ it is easy to see that

$$x \in \mathscr{M}^+, \varphi(x) = 0 \Rightarrow \varphi(\mathbf{s}(x)) = 0. \tag{1}$$

If $e, f \in \mathscr{M}$ are projections and $\varphi(e) = \varphi(f) = 0$, then $\varphi(e \vee f) = \varphi(\mathbf{s}(e + f)) = 0$. Thus the family $\mathscr{E} = \{e \in \operatorname{Proj}(\mathscr{M}); \varphi(e) = 0\}$ is upward directed. Let $e_0 = \sup \mathscr{E}$. By the normality of φ, it follows that $\varphi(e_0) = 0$, so that e_0 is the greatest projection in $\mathscr{M}$ annihilated by φ. The projection $\mathbf{s}(\varphi) = 1 - e_0$ is called the *support* of φ. Using (1), we obtain

$$\varphi(x^*x) = 0 \Leftrightarrow x\mathbf{s}(\varphi) = 0 \quad (x \in \mathscr{M}). \tag{2}$$

In particular, φ is faithful (1.1) if and only if $\mathbf{s}(\varphi) = 1$. Also

$$\varphi(x) = \varphi(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) \quad (x \in \mathscr{M}^+). \tag{3}$$

On the other hand, the w-closure $\overline{\mathfrak{N}_\varphi}^w$ of $\mathfrak{N}_\varphi$ is a w-closed left ideal of $\mathscr{M}$, hence $\overline{\mathscr{R}_\varphi}^w = \mathscr{M}e$ for some projection $e \in \mathscr{M}$ and $\overline{\mathfrak{M}_\varphi}^w = e\mathscr{M}e$ ([L], 3.20, 3.21). The weight φ is called *semifinite* if $e = 1$, that is, if $\mathfrak{N}_\varphi$, or equivalently, $\mathfrak{M}_\varphi$, is w-dense in $\mathscr{M}$. In this case, there exists an increasing net $\{u_i\}_{i\in I}$ in $\mathfrak{F}_\varphi = \mathfrak{M}_\varphi \cap \mathscr{M}^+$ such that $u_i \uparrow 1$ ([L], 3.20, 3.21).

We abbreviate the words “normal semifinite faithful” to *n.s.f.* Recall that on every W^*-algebra there exists an n.s.f. weight, while the countably decomposable W^*-algebras are characterized by the existence of a normal faithful positive form ([L], 10.14, E.5.6).

2.2 Theorem. *Let φ be a normal weight on the W^*-algebra $\mathscr{M}$. Then the associated GNS representation $\pi_\varphi : \mathscr{M} \to \mathscr{B}(\mathscr{H}_\varphi)$ is normal and nondegenerate. If φ is semifinite, then*

$$((\mathfrak{M}_\varphi)^n)_\varphi \text{ is dense in } \mathscr{H}_\varphi \quad (n \in \mathbb{N}). \tag{1}$$

*If φ is an n.s.f. weight, then π_φ is a *-isomorphism of $\mathscr{M}$ onto the von Neumann algebra $\pi_\varphi(\mathscr{M}) \subset \mathscr{B}(\mathscr{H}_\varphi)$.*

Proof. Clearly, $\pi_\varphi(1) = 1$, hence π_φ is nondegenerate. To show that π_φ is normal, that is, w-continuous, we have to check that $\omega \circ \pi_\varphi \in \mathscr{M}_*$ for every $\omega \in \mathscr{B}(\mathscr{H}_\varphi)_*$. Since the vector forms are total in $\mathscr{B}(\mathscr{H}_\varphi)_*$ ([L], 1.3) and $\mathfrak{N}_\varphi$ is dense in $\mathscr{H}_\varphi$, it is sufficient to do this only for $\omega = \omega_{a_\varphi, b_\varphi}$ with $a, b \in \mathfrak{N}_\varphi$. In this case, we have $\omega_{a_\varphi, b_\varphi} \circ \pi_\varphi = \varphi(b^* \cdot a) \in \mathscr{M}_*$, by Proposition 1.14. Since π_φ is normal and nondegenerate, $\pi_\varphi(\mathscr{M}) \subset \mathscr{B}(\mathscr{H}_\varphi)$ is a von Neumann algebra ([L], 3.12).

If φ is semifinite, then there exists an increasing net $\{u_i\}_{i\in I}$ in $\mathfrak{F}_\varphi = \mathfrak{M}_\varphi \cap \mathscr{M}^+$ with $u_i \uparrow 1$. For $a \in \mathfrak{N}_\varphi$, we have

$$\|a_\varphi - (u_i a)_\varphi\|_\varphi^2 = \varphi((a - u_i a)^*(a - u_i a)) \leq 2[\varphi(a^*a) - \varphi(a^*u_i a)] \to 0. \tag{2}$$