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Geometric Regular Polytopes

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Foreword

As might have been hoped, if not actually expected, since the publication of the monograph *Abstract Regular Polytopes* [99] by McMullen and Schulte there have been considerable advances in the subject. However, despite the obvious appeal of the geometric side of the theory, rather little space was devoted to it there. Indeed, the only systematic classification problems addressed in the book were those of the classical regular polytopes and honeycombs of Coxeter's seminal work *Regular Polytopes* [27], and what occurred in at most three dimensions. Otherwise, only sporadic examples were considered, such as ones illustrating aspects of realization theory or of regular polytopes whose universality is based on geometric constructions. In a sense, then, this book attempts to redress what might be perceived as an imbalance.

As we have just said, up to the publication of [99], the only systematic dimension-by-dimension investigation of regular polytopes had concentrated on small dimensions (at most three). In a sequence of subsequent papers [82, 83, 84, 86], McMullen has extended such classifications in several ways, by restricting attention to regular polytopes of full or nearly full rank (terms which will be defined in the text – the paper [98] by McMullen and Schulte can be regarded as the first of the sequence). To give examples, the subjects of [27] are the classical regular polytopes of full rank, though not all the regular polytopes and apeirotopes (that is, infinite polytopes) of full rank are classical.

It has therefore seemed appropriate to consolidate this line of research, and – with some rethinking of the basics – attempt to present it in a coherent way. Our main agenda are simple: to classify the regular polytopes and apeirotopes of full or nearly full rank. However, we shall also look at other interesting families that arose out of our investigations.

Unlike all Gaul, the book will be divided into four parts. Part I will cover those aspects of the abstract theory which we need subsequently, while Parts II and III will treat the cases of full and nearly full rank, respectively. Part IV contains somewhat of a miscellany. As we have indicated, it is our intention to recast the previously published material where we feel this to be necessary; we also take the opportunity to correct a number of mistakes in earlier treatments and repair some omissions.

In more detail, in Part I we describe the background to the theory of regular polytopes, both abstract and geometric. Chapter 1 sets the scene, by looking at some geometric foundations; in particular, it treats the discrete hyperplane reflexion groups, which play a central rôle in the whole book. In Chapter 2, we outline the abstract theory; this is a brief epitome of those parts of [99, Chapters 2–4] which have not been treated in Chapter 1. The geometric aspect is introduced in Chapters 3 and 4 through the theory of realizations; this material has been completely reworked from the earlier account in [99], and incorporates that from the recent new [85, 88], as well as some corrections of the theory by Ladisch [67]. Indeed, some concepts which were used in the earlier accounts have now disappeared, since they have proved redundant; conversely, more recent ideas – including some not previously in print – have resulted in a theory with considerable power. In Chapter 5 we consider various abstract and geometric operations and constructions on regular polytopes and apeirotopes. Finally, Chapter 6 introduces from [89] another recent concept, that of rigidity of regular polytopes; the basic question here asks to what extent the geometry of a regular polytope is determined by restricted geometric data.

There are two chapters in Part II. Chapter 7 successively covers the classical regular polytopes – including the star-polytopes – and apeirotopes, including an approach to the enumeration which is subsequent to that by Coxeter in [27]. One feature is the complete description in Section 7K of the realization domain of the 120-cell, which expands on the treatment in [92]. To a considerable extent (for the 4-dimensional polytopes) we follow the treatment of Du Val in the use of quaternions. The non-classical examples of full rank are covered in Chapter 8; a core feature of the classification is a restriction on the dimensions of the mirrors of their generating reflexions established in Section 4B.

Part III on the cases of nearly full rank is the longest, with five chapters. Attention must be drawn to the fact that these contain many corrections and additions to the papers on which they are based. Chapter 9 begins by treating the cases where blending is involved, and then classifying the various families which occur in each dimension. The remaining chapters deal with the pure polytopes and apeirotopes. Chapter 10 looks at the 3-dimensional apeirohedra; the material is extracted from [98], which was reproduced in [99, Section 7F], but we have augmented it by a discussion of the rigidity of the apeirohedra. (Indeed, wherever appropriate we say whether polytopes under consideration are rigid.) Chapter 11 then deals with the 4-dimensional polyhedra, expanding [83] a little, and with an excursion into an interesting family related to quasi-regular 3-dimensional polyhedra. The 4-dimensional apeirotopes of rank 4 are described in Chapter 12, which has been the most substantially reworked from the original paper [84]. Finally, following [86], Chapter 13 treats the regular polytopes and apeirotopes of nearly full rank in all higher dimensions.

The material in Part IV is hitherto unpublished. It illustrates further aspects of the foregoing theory, particularly those of realizations. In Chapter 14, we look more closely at the Gosset-Elte polytopes which are intimately related to several families of polytopes of nearly full rank; this provides a range of ways of applying realization theory, as well as being (we think) of intrinsic interest. The next Chapter 15 describes the realization domains of some of the locally toroidal polytopes that featured prominently in [99]; it also introduces a new class of universal such polytopes. Chapter 16 treats a family of 4-polytopes that displays some remarkable parallels with the pentagonal 4-polytopes of Chapter 7, while Chapter 17 deals with a family of 5-polytopes that combines both families of 4-polytopes as facets or vertex-figures. In part, as well, these two chapters provide further examples of the techniques used to determine realizations.

One feature of our treatment deserves special mention. It has always been our philosophy that one should try to approach any aspect of the subject using techniques which are as elementary as possible. This shows itself in various ways; for instance, in Section 1E we classify the finite and euclidean reflexion groups without appealing in any serious way to the theory of quadratic forms, as well as reproducing from [99, Section 3E] the earlier calculations of the finite orders using a convexity argument. Similarly, in Section 7E we find the Petrie polygons of the regular 600-cell and its relatives without solving trigonometric equations, while in Section 7G we determine the abstract groups of the 4-dimensional regular star-polytopes without going through the laborious changes of generators of [81] or [99, Theorem 7D16]. Even more striking, perhaps, though there are clear parallels between representations of finite groups and realizations of regular polytopes, particularly in orthogonality relations, we make almost no appeal to representation theory in our treatment of realizations.

It is inevitable that there is some overlap with [99]. For completeness of the exposition we need to cover a certain part of the introductory material, and the two accounts touch at various other places. However, even when we are not looking at things in a different way, for instance by adopting a more intuitive (but equivalent) definition of abstract polytopes, as we have indicated we have often reworked what we have written in various papers, as well as making some necessary corrections of them. Moreover, we have occasionally added background material that was omitted in [99]; for example, in Section 1G we give proofs of the Brianchon–Gram and Somerville theorems that lie behind the calculations of the orders of the finite Coxeter groups in Section 1H.

Several items in the bibilography are included for further reading, and are not cited in the text.

We have been encouraged to produce this book by friends and colleagues, of whom particular mention should be made to Asia Ivić Weiss, Barry Monson, Egon Schulte and Marston Conder. Apologies are also due to them for the time the book has taken to appear. Thanks are also due to the staff of Cambridge University Press, for their help and forbearance in getting this book into print.

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Frontmatter
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