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Euclidean Space

The main purpose of this chapter is to discuss groups generated by reflexions, concentrating here on the finite and discrete ones in euclidean spaces. There are several reasons for this. One rather important one is that this topic does not depend on anything that follows; indeed, to the contrary, we shall constantly appeal to reflexion groups for examples to illustrate the subsequent theory. In fact, until Part IV all but a single family of the regular polytopes described in this monograph have symmetry groups which are closely related to reflexion groups (if they are not reflexion groups themselves, then they are subgroups of them, or are obtained by twisting them with automorphisms). Moreover, it turns out that, except for one family, all finite or discrete affine reflexion groups are the symmetry groups of some regular polytopes or apeirotopes, even those which do not have linear diagrams.

The chapter contains ten sections. There are four preliminary ones, mainly to establish notation and conventions. Section 1A surveys the algebraic properties of euclidean spaces, while Sections 1C and 1D look at their metrical properties; in between, Section 1B covers the main features of convex sets that we shall need to appeal to. In the core Section 1E we classify the finite and discrete infinite reflexion groups in euclidean spaces; the initial part of our treatment is novel. In the next Section 1F we briefly comment on subgroup relationships among these groups. We also need to know the orders of the finite Coxeter groups; we find these by purely elementary geometric methods in Section 1H using angle-sum relations established in the previous Section 1G. The lower-dimensional spaces are somewhat special. For the following section, we need to know what the finite rotation groups in $\mathbb{E}^3$ are; this problem is solved in Section 1J. In 4-dimensional space $\mathbb{E}^4$, quaternions provide an alternative approach to finite orthogonal groups, and are actually needed to describe certain regular polyhedra in that space; what we want is covered in Section 1K.

We should emphasize that, by and large, we will only prove assertions made in this chapter if we need to employ them subsequently. Thus we shall include certain peripheral material as background, but not go into it in any great detail. And, of course, we shall try not to insult the reader by proving too many standard results in algebra and analysis.

1A Algebraic Properties

In this section, we are mainly interested in $\mathbb{E}^d$ as a linear (vector) or affine space; the extra properties induced by the inner product and norm will be discussed in Section 1C. As we said in the preamble to the chapter, a main purpose of this and the next section is to establish notation and conventions.

Linear Spaces

For the moment, therefore, we just consider finite dimensional real linear (or vector) spaces. Indeed, in this section, only the fact that the real numbers $\mathbb{R}$ form a field is material; at this stage, it is not important that $\mathbb{R}$ is ordered. Thus, $\mathbb{X}$, $\mathbb{Y}$, and so on, will be finite dimensional linear spaces over $\mathbb{R}$.

We assume that the reader is familiar with the fundamental algebraic ideas of groups, rings, fields and linear spaces. In particular, in the latter context, the notions of linear combination, linear dependence and independence, linear subspace and (linear) basis will be taken for granted. The only point we wish to make here is notational: the *linear hull* $\operatorname{lin} X$ of $X \subseteq \mathbb{X}$ is

- the set of linear combinations of vectors in X,
- the intersection of the linear subspaces of $\mathbb{X}$ which contain X.

By definition, the zero vector (or origin) $o \in \operatorname{lin} X$ always.

The basic operations of a linear space extend to subsets. Thus, for $X, Y \subseteq \mathbb{X}$ and $\lambda \in \mathbb{R}$, we define the *sum* $X + Y$ and *scalar multiple* λX by

1A1 $$X + Y := \{x + y \mid x \in X,\ y \in Y\},$$

1A2 $$\lambda X := \{\lambda x \mid x \in X\}.$$

In particular, we write $X + t := X + \{t\}$ for the *translate* of X by $t \in \mathbb{X}$; then t is called the corresponding *translation vector*.

Affine Properties

In some contexts, though, it is inconvenient to have the zero vector o playing a special rôle, and so it is preferable to regard $\mathbb{X}$ as an affine space. The *line* xy through $x, y \in \mathbb{X}$ is

1A3 $$xy := \{(1-\lambda)x + \lambda y \mid \lambda \in \mathbb{R}\} \subseteq A;$$

an *affine subspace* A in $\mathbb{X}$ is determined by the fact that, if $x, y \in A$ then $xy \subseteq A$ (see the notes at the end of the section). Actually, the definition allows the empty set $\emptyset$ and point-sets to be affine subspaces as well (note that $xx = \{x\}$); contrast the former with the fact that *linear* subspaces always contain o, and so are non-empty. An easy exercise shows the following. Define the *affine hull* $\operatorname{aff} X$ of a subset $X \subseteq \mathbb{X}$ by

1A4 $$\operatorname{aff} X := \bigcap \{A \subseteq \mathbb{X} \mid A \text{ an affine subspace, and } X \subseteq A\}.$$

Then aff X consists of all *affine* combinations

1A5 $$\lambda_0 x_0 + \cdots + \lambda_k x_k, \qquad \lambda_0 + \cdots + \lambda_k = 1,$$

of points $x_0, \ldots, x_k \in X$. We also say that A *spans* aff A *affinely*. Moreover, we have

1A6 Proposition *A non-empty affine subspace A is a translate $A = L + t$ of a (unique) linear subspace L.*

Proof. Indeed, if $t \in A$ is any point, then it is straightforward to show that $L := A - t$ is a linear subspace. Observe that, if $t' \in A$ also, then $t' - t \in L$, so that

$$A - t' = (A - t) - (t' - t) = L - (t' - t) = L;$$

the uniqueness of L is a consequence. □

We say that two affine subspaces A_1, A_2 are *parallel* if A_2 is a translate of A_1; hence, parallel affine subspaces are translates of the same linear subspace.

The concepts of affine dependence, independence and basis are the natural extensions of the linear notions. Thus an *affinely independent* set B is such that no one of its points is an affine combination of the others. Equivalently, $B = \{b_0, \ldots, b_k\}$ is affinely independent if $\xi_0 b_0 + \cdots + \xi_k b_k = o$ for $\xi_0, \ldots, \xi_k \in \mathbb{R}$ such that $\xi_0 + \cdots + \xi_k = 0$ implies that $\xi_0 = \cdots = \xi_k = 0$. An *affine basis* of $\mathbb{X}$ is an affinely independent set $B \subseteq \mathbb{X}$ which spans $\mathbb{X}$ affinely.

1A7 Proposition *An affine basis of a d-dimensional space $\mathbb{X}$ consists of $d+1$ points.*

Proof. It is easily shown that $\{b_0, \ldots, b_d\}$ is affinely independent if and only if $\{b_1 - b_0, \ldots, b_d - b_0\}$ is linearly independent; the claim then follows. □

The obvious definition of the *dimension* dim A of an affine subspace A is $\dim A := \dim L$, if $A = L + t$ for some linear subspace L. Thus an affine basis of A has $\dim A + 1$ points; compare Proposition 1A7. For the empty set, the natural definition is thus $\dim \emptyset := -1$. Often also useful is the notion of *codimension* $\operatorname{codim} A := \dim \mathbb{X} - \dim A$. In particular, an affine subspace of codimension 1 is called a *hyperplane*.

1A8 Remark In some contexts, like those of realizations (see, for example, Section 3L) we find it useful to work in linear spaces over ordered fields other than $\mathbb{R}$. Of particular interest are the *rational numbers* $\mathbb{Q}$ and (real) *algebraic numbers* $\mathbb{A}$.

Mappings

We next look at mappings. Again, we assume that the reader is familiar with linear mappings; however, we wish to recall some terminology and introduce some notation.

If $\Phi\colon \mathbb{X} \to \mathbb{Y}$ is a linear mapping, then we denote by $\operatorname{im}\Phi$ its *image space* and by $\ker\Phi$ its *kernel*; their dimensions are the *rank* $\operatorname{rank}\Phi$ and *nullity* $\operatorname{null}\Phi$, respectively. Recall that the latter are related by

$$\operatorname{rank}\Phi + \operatorname{null}\Phi = \dim\mathbb{X}.$$

For fixed $\mathbb{X}$ and $\mathbb{Y}$, the family of linear mappings $\Phi\colon \mathbb{X} \to \mathbb{Y}$ forms, in a natural way, a linear space $\operatorname{Hom}(\mathbb{X},\mathbb{Y})$ of dimension $\dim\mathbb{X}\dim\mathbb{Y}$.

A linear mapping $u\colon \mathbb{X} \to \mathbb{R}$ is called a *linear functional*; in this special case, we have the *dual space* $\mathbb{X}^* := \operatorname{Hom}(\mathbb{X},\mathbb{R})$. There is a natural pairing $\langle\cdot,\cdot\rangle$ on $\mathbb{X}\times\mathbb{X}^*$, so that we write the image of $x \in \mathbb{X}$ under $u \in \mathbb{X}^*$ as $\langle x,u\rangle = \langle u,x\rangle$, thus emphasizing the underlying symmetry. Corresponding to a basis $E = (e_1,\dots,e_d)$ of $\mathbb{X}$ is a *dual basis* $E^* = (e_1^*,\dots,e_d^*)$ of $\mathbb{X}^*$, which satisfies

$$\langle e_j, e_k^*\rangle = \delta_{jk} := \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases}$$

the Kronecker delta.

If $x \in \mathbb{X}$, then writing $x = \sum_{j=1}^{d} \xi_j e_j$ and applying e_k^* shows that $\xi_k = \langle x, e_k^*\rangle$. In other words, if $x \in \mathbb{X}$, then

1A9
$$x = \sum_{j=1}^{d} \langle x, e_j^*\rangle e_j.$$

Another familiar fact is

1A10 Proposition *A hyperplane H in $\mathbb{X}$ can be represented as $H(u,\beta) := \{x \in \mathbb{X} \mid \langle x,u\rangle = \beta\}$, for some non-zero $u \in \mathbb{X}^*$ and $\beta \in \mathbb{R}$.*

Proof. By Proposition 1A6, H is a translate $H = H_0 + t$ of a linear hyperplane H_0 of $\mathbb{X}$. If $\dim\mathbb{X} = d$, choose any (linear) basis $\{a_1,\dots,a_{d-1}\}$ of H_0, and extend to a basis $\{a_1,\dots,a_d\}$ of $\mathbb{X}$. If $\{a_1^*,\dots,a_d^*\}$ is the dual basis of $\mathbb{X}^*$, then we define $u := a_d^*$, so that $H_0 = \{x \in \mathbb{X} \mid \langle x,u\rangle = 0\}$. It follows at once that $H = H(u,\beta)$, with $\beta := \langle t,u\rangle$, which is as asserted. □

For now, we only need the idea of an *affine mapping* $\Phi\colon \mathbb{X} \to \mathbb{Y}$, with $\mathbb{X},\mathbb{Y}$ finite dimensional real linear spaces: this is such that

1A11
$$((1-\lambda)x + \lambda y)\Phi = (1-\lambda)x\Phi + \lambda y\Phi$$

for all $x, y \in \mathbb{X}$ and $\lambda \in \mathbb{R}$. A straightforward inductive argument shows that affine mappings preserve arbitrary affine combinations. Moreover, we actually have

1A12 Proposition *If $\Phi\colon \mathbb{X} \to \mathbb{Y}$ is an affine mapping, then there is a $t \in \mathbb{Y}$ and a linear mapping $\Psi\colon \mathbb{X} \to \mathbb{Y}$ such that $x\Phi = x\Psi + t$ for all $x \in \mathbb{X}$.*

Proof. Define $t := o\Phi$ and Ψ by $x\Psi := x\Phi - t$. We leave to the reader the easy exercise of completing the proof (that is, showing that Ψ is linear). □

An invertible affine mapping $\Phi\colon \mathbb{X} \to \mathbb{X}$ is also called an *affinity*; clearly Φ is invertible just when the associated linear mapping Ψ of Proposition 1A12 is invertible. As a particular affinity, we have the *translation* $x \mapsto x+t$, with $t \in \mathbb{X}$ as before the corresponding translation vector.

Matrices

As is well known, a linear mapping $\Phi\colon \mathbb{X} \to \mathbb{Y}$ can be represented by a matrix with respect to a choice of bases of $\mathbb{X}$ and $\mathbb{Y}$; this matrix will be invertible just when Φ is invertible.

We shall often associate an ordered set of vectors $(a_1, \ldots, a_k)$ in $\mathbb{R}^m$ with the $k \times m$ matrix A whose rows are the a_i. If B is an $m \times n$ matrix, then it is sometimes useful to think of the entries of the product AB as $\langle a_i, b_j \rangle$, where $b_1, \ldots, b_m$ are now the columns of B, regarded as vectors in the dual space $(\mathbb{R}^m)^*$.

Recall that the *trace* $\operatorname{tr} A$ of an $m \times n$ matrix $A = (\alpha_{ij})$ is

1A13 $$\operatorname{tr} A := \sum_j \alpha_{jj},$$

the range of summation being $1 \leqslant j \leqslant \min\{m, n\}$. Thus $\operatorname{tr} A = \operatorname{tr} A^{\mathsf{T}}$, with $A^{\mathsf{T}} = (\beta_{ij})$ the *transpose* of A, so that $\beta_{ij} = \alpha_{ji}$ for each i, j. Moreover, if B is an $n \times m$ matrix, then we have

1A14 $$\operatorname{tr}(AB) = \operatorname{tr}(BA),$$

as is easy to see.

If the linear mapping $\Phi\colon \mathbb{X} \to \mathbb{Y}$ is represented by the matrix A with respect to given bases of $\mathbb{X}$ and $\mathbb{Y}$, then the dual mapping $\Phi^*\colon \mathbb{Y}^* \to \mathbb{X}^*$ is represented by the transpose matrix A^{T} with respect to the dual bases of $\mathbb{Y}^*$ and $\mathbb{X}^*$.

Groups

A finite group $\mathbf{G}$ of affinities on $\mathbb{X}$ with order $|\mathbf{G}| := \operatorname{card} \mathbf{G}$ has a fixed point, namely, the centroid

$$c := \frac{1}{|\mathbf{G}|} \sum_{\Phi \in \mathbf{G}} x\Phi$$

of the images of an arbitrary point $x \in \mathbb{X}$ under $\mathbf{G}$ (there may be more than one such point c). Conjugating $\mathbf{G}$ by a translation which takes c to the origin shows that we lose no generality in assuming that $\mathbf{G}$ is a subgroup of the *general linear group* $\mathbf{GL}(\mathbb{X}) := \operatorname{Hom}(\mathbb{X}, \mathbb{X})$ of invertible *linear* mappings on $\mathbb{X}$.

We say that $\mathbb{L} \leqslant \mathbb{X}$ is an *invariant subspace* of $\mathbf{G}$ if $x\Phi \in \mathbb{L}$ for all $x \in \mathbb{L}$ and $\Phi \in \mathbf{G}$. We call $\mathbf{G}$ *irreducible* if its only invariant subspaces are $\{o\}$ and $\mathbb{X}$ itself.

We call two subgroups $\mathbf{G}, \mathbf{H} \leqslant \mathbf{GL}(\mathbb{X})$ of linear mappings *linearly equivalent* if they are conjugate under some $\Theta \in \mathbf{GL}(\mathbb{X})$, so that $\mathbf{H} = \Theta^{-1}\mathbf{G}\Theta$. Observe that, if $\mathbf{G}$ is irreducible, then so is $\mathbf{H}$.

Direct Sums

Linear spaces $\mathbb{X}$ and $\mathbb{Y}$ can be combined in two ways (for our purposes). First, we have the ordinary *direct sum* or *cartesian product* $\mathbb{X} \oplus \mathbb{Y}$. The usual way of expressing a vector $z \in \mathbb{X} \oplus \mathbb{Y}$ is as $z = (x, y)$; this is particularly appropriate when x and y are coordinate vectors (with respect to chosen bases of the two spaces). We thus have $\dim(\mathbb{X} \oplus \mathbb{Y}) = \dim \mathbb{X} + \dim \mathbb{Y}$.

1A15 Remark It is frequently the case that we express a given linear space as an *internal* direct sum of subspaces $\mathbb{L}_1, \mathbb{L}_2, \ldots$, and so write

$$\mathbb{X} = \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \cdots .$$

Rings and Algebras

When we come to realizations in Chapter 3, we shall want to phrase things in terms of direct sums of rings or, rather, of algebras. An *algebra* $\mathbb{K}$ over $\mathbb{R}$ (this is all we need) is a real vector space for addition and scalar multiplication; further, it has an associative multiplication which distributes over addition and scalar multiplication, and so satisfies

$$\begin{aligned} a(b + c) &= ab + ac, \\ (a + b)c &= ac + bc, \\ (\lambda a)b = a(\lambda b) &= \lambda(ab), \end{aligned}$$

for all $a, b, c \in \mathbb{K}$ and $\lambda \in \mathbb{R}$. The *direct sum* $\mathbb{K}_1 \oplus \mathbb{K}_2$ of two algebras $\mathbb{K}_1$ and $\mathbb{K}_2$ then just satisfies the algebra properties in each coordinate separately, so that the basic operations are

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, a_2 + b_2), \\ \lambda(a_1, a_2) &= (\lambda a_1, \lambda a_2), \\ (a_1, a_2)(b_1, b_2) &= (a_1 b_1, a_2 b_2). \end{aligned}$$

For further properties of direct sums of rings, see for example [4, Chapter 13] or [64, Chapter 1].

Tensor Products

The other combination is the *tensor product* $\mathbb{X} \otimes \mathbb{Y}$, an element of which is called a *tensor*. Formally, this is the *universal* linear space $\mathbb{W}$ for mappings Φ on $\mathbb{X} \times \mathbb{Y}$ that are *bilinear*, in that

1A16 $$(\lambda x + \mu z, y)\Phi = \lambda(x, y)\Phi + \mu(z, y)\Phi,$$

with symmetric expressions for the second term, and which is such that any bilinear mapping Φ on $\mathbb{X} \times \mathbb{Y}$ induces a linear mapping on $\mathbb{W}$. We need to know two things about the tensor product. First, $\dim(\mathbb{X} \otimes \mathbb{Y}) = \dim \mathbb{X} \dim \mathbb{Y}$: if

$\{f_1, \ldots, f_d\}$ is a (linear) basis of $\mathbb{X}$ and $\{g_1, \ldots, g_c\}$ is one of $\mathbb{Y}$, then $e_{ij} := f_i \otimes g_j$ gives a basis of $\mathbb{X} \otimes \mathbb{Y}$. Second, linear mappings Φ on $\mathbb{X}$ and Ψ on $\mathbb{Y}$ induce a linear mapping $\Phi \otimes \Psi$ on $\mathbb{X} \otimes \mathbb{Y}$ by

1A17 $$(x \otimes y)(\Phi \otimes \Psi) := (x\Phi) \otimes (y\Psi).$$

In our treatment, we never have to deal with other than a *simple* tensor $x \otimes y$, with $x \in \mathbb{X}$ and $y \in \mathbb{Y}$.

1A18 Remark There is a natural isomorphism $\operatorname{Hom}(\mathbb{X}, \mathbb{Y}) \cong \mathbb{X}^* \otimes \mathbb{Y}$. Observe that, *up to isomorphism*, the tensor product is associative and commutative.

An element φ of the dual space $\mathbb{X}^* \otimes \mathbb{Y}^*$ of $\mathbb{X} \otimes \mathbb{Y}$ is also called a *bilinear form*. If $\mathbb{X} = \mathbb{Y}$ and $\varphi(x, y) = \varphi(y, x)$ for all $x, y \in \mathbb{X}$, then we call the form *symmetric*. The case $x = y$ gives a *quadratic form*. We say that φ is *positive semi-definite* if $\varphi(x, x) \geqslant 0$ for all $x \in \mathbb{E}^d$, and *positive definite* if $\varphi(x, x) > 0$ whenever $x \neq o$. If $\dim \mathbb{X} = d$, then we can write a quadratic form as $\varphi(x, x) = xAx^{\mathsf{T}}$, with now x regarded as a coordinate vector with respect to a chosen basis of $\mathbb{X}$ and $A = (\alpha_{jk})$ a $d \times d$ *symmetric* matrix, meaning that $\alpha_{jk} = \alpha_{kj}$ for all $j, k = 1, \ldots, d$.

When $\mathbb{X} = \mathbb{Y}$, we have refinements of the tensor product. In the space of r-fold tensors $x_1 \otimes \cdots \otimes x_r$, we can make two kinds of identification. First, taking $x_j \otimes x_{j+1} = x_{j+1} \otimes x_j$ (for $j = 1, \ldots, r-1$) gives a *symmetric* tensor. In this case, we can write $x_j \otimes x_{j+1} =: x_j x_{j+1}$ as an ordinary product. Second, setting $x_j \otimes x_{j+1} + x_{j+1} \otimes x_j = o$ gives an *alternating* tensor; here, it is important that we impose identifications on *adjacent* terms in an alternating tensor product $x_1 \wedge \cdots \wedge x_k$, so that $x_{j+1} \wedge x_j = -(x_j \wedge x_{j+1})$. Both notions occur in the theory of realizations, but we shall see that alternating tensors only occasionally play a useful rôle.

A linear mapping Φ on $\mathbb{X}$ induces a linear mapping $\wedge^r \Phi$ on $\wedge^r \mathbb{X}$. If $\dim \mathbb{X} = d$, then $\wedge^d \mathbb{X} \cong \mathbb{R}$, so that, if $\Phi \colon \mathbb{X} \to \mathbb{X}$, then $\wedge^d \Phi \colon \mathbb{R} \to \mathbb{R}$ is just multiplication by a scalar, which is denoted $\det \Phi$ and called the *determinant* of Φ.

In a similar way, a $d \times d$ matrix A induces a linear mapping on $\mathbb{X}$ with respect to a chosen basis $(e_1, \ldots, e_d)$, say. Then $\det A$ is independent of the basis, and is naturally called the determinant of A. We can identify $a_j = e_j A$ with the jth row of A. The determinant shows that there is a natural identification of $(d-1)$-fold alternating tensors with linear functionals on $\mathbb{X}$, namely,

$$\langle x, a_2 \wedge \cdots \wedge a_d \rangle = \det(x, a_2, \ldots, a_d).$$

With the single exception $d = 3$, this is of little general interest here; the exception involves the definition of quaternions in Section 1K.

Notes to Section 1A

1. Our convention, in which we follow Coxeter [27], is to write mappings *after* their arguments. We rarely have compositions of mappings except in the context of

groups; here, it seems to us more natural to compose mappings in the order in which they are applied. Moreover – and this is useful in another way – we can then think of vectors as *rows* rather than columns, which obviates the need for any special conventions in text.

2. Affine subspaces are also known as *flats*. We do not use the term in this context, because later on we encounter a quite different concept of flatness.
3. A hint to Proposition 1A12 is to note that $\lambda x + \mu y = \lambda x + \mu y + (1 - \lambda - \mu)o$, as an affine combination.
4. Observe that, for dual spaces, we have $(\mathbb{X} \otimes \mathbb{Y})^* = \mathbb{X}^* \otimes \mathbb{Y}^*$.
5. Similarly, a linear mapping $\Phi\colon \mathbb{X} \to \mathbb{Y}$ induces a dual linear mapping $\Phi^*\colon \mathbb{Y}^* \to \mathbb{X}^*$, defined by $\langle x, v\Phi^*\rangle := \langle x\Phi, v\rangle$ for all $x \in \mathbb{X}$ and $v \in \mathbb{Y}^*$. But we make no future use of this or the previous concept.

1B Convexity

From now on, we use the fact that $\mathbb{R}$ is an *ordered* field, and so it should not be surprising that concepts such as positivity come into play. We shall make extensive use of this and the closely related concept of convexity in various places in the book.

Positive Combinations

For the analogue of 'linear', we call $C \subseteq \mathbb{X}$ a (*convex*) *cone* if $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \geqslant 0$. Thus the *positive hull* of $X \subseteq \mathbb{X}$ is

1B1 $$\operatorname{pos} X := \bigcap\{C \subseteq \mathbb{X} \mid C \text{ a cone, and } X \subseteq C\}.$$

Then $\operatorname{pos} X$ consists of all *positive combinations*

1B2 $$\lambda_1 x_1 + \cdots + \lambda_k x_k, \qquad \lambda_1, \ldots, \lambda_k \geqslant 0,$$

of points $x_1, \ldots, x_k \in X$; we also say that $\operatorname{pos} X$ *spans* X *positively*.

As a special case, if X is linearly independent, then we call $K := \operatorname{pos} X$ a *simple* cone. If C is a cone of dimension $\dim C := \dim \operatorname{lin} C = d$, then we refer to C as a d-*cone*.

In a couple of places we shall need an important result from convexity. We state the theorem first as it applies to cones. A *ray* $[oa$ (or *half-line*) is a set of the form

1B3 $$[oa := \{\lambda a \mid \lambda \geqslant 0\},$$

for some $a \in \mathbb{E}^d$. If C is a closed convex cone, then a ray $E \subseteq C$ is called *extreme* if, whenever $a \in E$ and $x, y \in E$ satisfy $a = x + y$, then x and y are scalar multiples of a.

We then have *Carathéodory's Theorem* (see the notes at the end of the section); we refer to (for example) [132, Proposition 1.15] for more details.

1B4 Theorem *If C is a closed convex cone of dimension d, then each point of C is a sum of at most d points on extreme rays of C.*