

Spacetime and Geometry

An Introduction to General Relativity

Spacetime and Geometry is an introductory textbook on general relativity, specifically aimed at students. Using a lucid style, Carroll first covers the foundations of the theory and mathematical formalism, providing an approachable introduction to what can often be an intimidating subject. Three major applications of general relativity are then discussed: black holes, perturbation theory and gravitational waves, and cosmology. Students will learn the origin of how spacetime curves (the Einstein equation) and how matter moves through it (the geodesic equation). They will learn what black holes really are, how gravitational waves are generated and detected, and the modern view of the expansion of the universe. A brief introduction to quantum field theory in curved spacetime is also included. A student familiar with this book will be ready to tackle research-level problems in gravitational physics.

Sean M. Carroll is Research Professor of Physics at the California Institute of Technology. His research focuses on general relativity, cosmology, field theory, statistical mechanics, and quantum mechanics. He is the recipient of numerous awards, including the Gemant Award from the American Institute of Physics, the Winton Science Book Prize from the Royal Society, a Guggenheim fellowship, and teaching awards from MIT and the University of Chicago.

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SEAN M. CARROLL



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“For if each Star is little more a mathematical Point, located upon the Hemisphere of Heaven by Right Ascension and Declination, then all the Stars, taken together, tho’ innumerable, must like any other set of points, in turn represent some single gigantic Equation, to the mind of God as straightforward as, say, the Equation of a Sphere,—to us unreadable, incalculable. A lonely, uncompensated, perhaps even impossible Task,—yet some of us must ever be seeking, I suppose.”

—Thomas Pynchon, *Mason & Dixon*

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Preface

General relativity is the most beautiful physical theory ever invented. It describes one of the most pervasive features of the world we experience—gravitation—in terms of an elegant mathematical structure—the differential geometry of curved spacetime—leading to unambiguous predictions that have received spectacular experimental confirmation. Consequences of general relativity, from the big bang to black holes, often get young people first interested in physics, and it is an unalloyed joy to finally reach the point in one’s studies where these phenomena may be understood at a rigorous quantitative level. If you are contemplating reading this book, that point is here.

In recent decades, general relativity (GR) has become an integral and indispensable part of modern physics. For a long time after it was proposed by Einstein in 1916, GR was counted as a shining achievement that lay somewhat outside the mainstream of interesting research. Increasingly, however, contemporary students in a variety of specialties are finding it necessary to study Einstein’s theory. In addition to being an active research area in its own right, GR is part of the standard syllabus for anyone interested in astrophysics, cosmology, string theory, and even particle physics. This is not to slight the more pragmatic uses of GR, including the workings of the Global Positioning System (GPS) satellite network.

There is no shortage of books on GR, and many of them are excellent. Indeed, approximately thirty years ago witnessed the appearance of no fewer than three books in the subject, each of which has become a classic in its own right: those by Weinberg (1972), Misner, Thorne, and Wheeler (1973), and Hawking and Ellis (1975). Each of these books is suffused with a strongly-held point of view advocated by the authors. This has led to a love-hate relationship between these works and their readers; in each case, it takes little effort to find students who will declare them to be the best textbook ever written, or other students who find them completely unpalatable. For the individuals in question, these judgments may very well be correct; there are many different ways to approach this subject.

The present book has a single purpose: to provide a clear introduction to general relativity, suitable for graduate students or advanced undergraduates. I have attempted to include enough material so that almost any one-semester introductory course on GR can find the appropriate subjects covered in the text, but not too much more than that. In particular, I have tried to resist the temptation to write a comprehensive reference book. The only goal of this book is to teach you GR. An intentional effort has been made to prefer the conventional over the idiosyncratic. If I can be accused of any particular ideological bias, it would be a

tendency to think of general relativity as a field theory, a point of view that helps one to appreciate the connections among GR, particle physics, and string theory. At the same time, there are a number of exciting astrophysical applications of GR (black holes, gravitational lensing, the production and detection of gravitational waves, the early universe, the late universe, the cosmological constant), and I have endeavored to include at least enough background discussion of these issues to prepare students to tackle the current literature.

The primary question facing any introductory treatment of general relativity is the level of mathematical rigor at which to operate. There is no uniquely proper solution, as different students will respond with different levels of understanding and enthusiasm to different approaches. Recognizing this, I have tried to provide something for everyone. I have not shied away from detailed formalism, but have also attempted to include concrete examples and informal discussion of the concepts under consideration. Much of the most mathematical material has been relegated to the Appendices. Some of the material in the Appendices is actually an integral part of the course (for example, the discussion of conformal diagrams), but an individual reader or instructor can decide just when it is appropriate to delve into them; signposts are included in the body of the text.

Surprisingly, there are very few formal prerequisites for learning general relativity; most of the material is developed as we go along. Certainly no prior exposure to Riemannian geometry is assumed, nor would it necessarily be helpful. It would be nice to have already studied some special relativity, although a discussion is included in Chapter 1, its purpose is more to review the basics and and introduce some notation, rather than to provide a self-contained introduction. Beyond that, some exposure to electromagnetism, Lagrangian mechanics, and linear algebra might be useful, but the essentials are included here.

The structure of the book should be clear. The first chapter is a review of special relativity and basic tensor algebra, including a brief discussion of classical field theory. The next two chapters introduce manifolds and curvature in some detail; some motivational physics is included, but building a mathematical framework is the primary goal. General relativity proper is introduced in Chapter 4, along with some discussion of alternative theories. The next four chapters discuss the three major applications of GR: black holes (two chapters), perturbation theory and gravitational waves, and cosmology. Each of these subjects has witnessed an explosion of research in recent years, so the discussions here will be necessarily introductory, but I have tried to emphasize issues of relevance to current work. These three applications can be covered in any order, although there are interdependencies highlighted in the text. Discussions of experimental tests are sprinkled through these chapters. Chapter 9 is a brief introduction to quantum field theory in curved spacetime; this is not a necessary part of a first look at GR, but has become increasingly important to work in quantum gravity and cosmology, and therefore deserves some mention. On the other hand, a few topics are scandalously neglected; the initial-value problem and cosmological perturbation theory come to mind, but there are others. Fortunately there is no shortage of other resources. The Appendices serve various purposes: There are discussions of

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technical points that were avoided in the body of the book, crucial concepts that could have been put in various places, and extra topics that are useful but outside the main development.

Since the goal of the book is pedagogy rather than originality, I have often leaned heavily on other books (listed in the bibliography) when their expositions seemed perfectly sensible to me. When this leaning was especially heavy, I have indicated it in the text itself. It will be clear that a primary resource was the book by Wald (1984), which has become a standard reference in the field; readers of this book will hopefully be well-prepared to jump into the more advanced sections of Wald's book.

This book grew out of a set of lecture notes that were prepared when I taught a course on GR at MIT. These notes are available on the web for free, and will continue to be so; they will be linked to the website listed below. Perhaps a little over half of the material here is contained in the notes, although the advantages of owning the book (several copies, even) should go without saying.

Countless people have contributed greatly both to my own understanding of general relativity and to this book in particular—too many to acknowledge with any hope of completeness. Some people, however, deserve special mention. Ted Pyne learned the subject along with me, taught me a great deal, and collaborated with me the first time we taught a GR course, as a seminar in the astronomy department at Harvard; parts of this book are based on our mutual notes. Nick Warner taught the course at MIT from which I first learned GR, and his lectures were certainly a very heavy influence on what appears here. Neil Cornish was kind enough to provide a wealth of exercises, many of which have been included at the end of each chapter. And among the many people who have read parts of the manuscript and offered suggestions, Samaz Arkani-Hamed was kind enough to go through the entire thing in great detail.

I would also like to thank everyone who either commented in person or by email on different parts of the book; these include Tigran Alvezian, Teodora Bereshka, Ed Bertschinger, Patrick Brady, Peter Brown, Jennifer Chen, Michele Ferraz Figueiró, Eanna Flanagan, Jacques Fric, Ygor Geurts, Marco Godina, Monica Guica, Jim Hartle, Tamás Hauer, Daniel Holz, Ted Jacobson, Akash Kansagra, Chuck Keeton, Arthur Kosowsky, Eugene Lim, Jorma Louko, Robert A. McNees, Hayri Mutluay, Simon Ross, Itai Seggev, Robert Wald, and Barton Zwiebach. Apologies are due to anyone I may have neglected to mention. And along the way I was fortunate to be the recipient of wisdom and perspective from numerous people, including Shadi Bartsch, George Field, Deryn Fogg, Ilana Har-rus, Gretchen Heitrich, Mari Ruti, Maria Spiropulu, Mark Trodden, and of course my family. (This wisdom often came in the form, "What were you thinking?") Finally, I would like to thank the students in my GR classes, on whom the strategies deployed here were first tested, and express my gratitude to my students and collaborators, for excusing my book-related absences when I should have been doing research.

My friends who have written textbooks themselves tell me that the first printing of a book will sometimes contain mistakes. In the unlikely event that this happens

Preface

here, there will be a list of errata kept at the website for the book:

<http://spacetimeandgeometry.net/>

The website will also contain other relevant links of interest to readers.

During the time I was working on this book, I was supported by the National Science Foundation, the Department of Energy, the Alfred P. Sloan Foundation, and the David and Lucile Packard Foundation.

Sean Carroll
Chicago, Illinois
June 2003

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