Mathematical Methods for Oscillations and Waves

Anchored in simple and familiar physics problems, Joel Franklin provides a focused introduction to mathematical methods in a narrative-driven and structured manner. Ordinary and partial differential equation solving, linear algebra, vector calculus, complex variables, and numerical methods are all introduced and bear relevance to a wide range of physical problems. Expanded and novel applications of these methods highlight their utility in less familiar areas, and advertise those areas that will become more important as students continue. This highlights both the utility of each method in progressing with problems of increasing complexity while also allowing students to see how a simplified problem becomes "recomplexified." Advanced topics include nonlinear partial differential equations, and relativistic and quantum mechanical variants of problems like the harmonic oscillator. Physics, mathematics, and engineering students will find 300 problems treated in a sophisticated manner. The insights emerging from Franklin's treatment make it a valuable teaching resource.

Joel Franklin is a professor in the Physics Department of Reed College. His research focuses on mathematical and computational methods with applications to classical mechanics, quantum mechanics, electrodynamics, general relativity, and modifications to general relativity. He is also the author of *Advanced Mechanics and General Relativity* (Cambridge University Press, 2010), *Computational Methods for Physics* (Cambridge University Press, 2013), and *Classical Field Theory* (Cambridge University Press, 2017).

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For Lancaster, Lewis, Oliver, and Mom

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Joel Franklin
Frontmatter
More Information

Contents

Preface

There are many books on "mathematical methods for physics" [1, 3, 15], including some with that exact title. Most of these are wide-ranging explorations of the physical applications of fairly deep analytic and group-theoretic mathematics. They cover topics that one might encounter anywhere from first-year undergraduate to first-year graduate physics, and remain on our shelves as well-thumbed references and problem libraries. In addition to a plethora of techniques, they cover all sorts of important special cases that can keep the naïve physicist out of trouble in a variety of technical situations.

There is also the Internet, itself a repository for all sorts of human knowledge, including physical, mathematical, and their intersection. The Internet is even more encyclopedic than most mathematical methods books, with more special cases and more specialized examples. Here we can find, in almost equal number, the inspiring, the arcane, and the incorrect. Students of physics, especially early in their studies, need to be sophisticated and wary.

What is missing in both cases (especially the latter) is narrative. A clear description of why we care about these methods, and how they are related to diverse, yet logically connected problems of interest to physicists. Why is it, for example, that the Fourier transform shows up in the analysis of networks of springs and also in the analysis of analog circuits? I suggest the reason is that both involve the characterization of timescales of oscillation and decay, and in some sense, almost all of physics is interested in such timescales, so there is a universality here that is not shared with, say, the Laplace transform. Yet Wikipedia, and other "online resources" fail to make this point – or rather point and counterpoint – effectively, because there is no individual curator deciding what goes in, what stays out, and how much time/space to dedicate to each topic.

This book has such a curator, and I have made choices based on my own research experience, broadened by the teaching I have done at Reed College, and feedback from students I have taught. At a small liberal arts college like Reed, the faculty must teach and advise students in areas beyond the faculty member's expertise and experience. The advantage is a generalist's view, but with the depth that is required to teach extremely curious and talented students confidently. After all, much of what we sell in physics is counterintuitive or even wrong (witness: gravity as one of the fundamental forces of nature). We should expect, delight in, and encourage our students' skepticism. The topics in this book are intended to invite students to ask and answer many of the questions I have been asked by their peers over the past 15 years.

In my department, there is a long tradition of teaching mathematical methods using oscillations and waves as a motivational topic. And there are many appropriate "oscillations and waves" texts and monographs [9, 16]. These are typically short supplemental books that exhaustively treat the topic. Yet few of them attempt to extend their reach fully, to include the mathematical methods that, for example, might be useful to a student of E&M (an exception is [10], which does have a broader mathematical base). I was inspired by Sidney Coleman's remark, "The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction." I am not sure any physics that I know is particularly far removed from the harmonic oscillator, and embracing that sentiment gives one plenty of room to maneuver. There is no reason that the mathematical methods of oscillations and waves can't serve as a stand-in for "mathematical methods for physics."

I have used chapters of the present volume to teach a one-semester undergraduate course on mathematical physics to second-year physics students. For that audience, I work through the following chapters:

Chapter 1 **Harmonic Oscillator**: A review of the problem of motion for masses attached to springs. That's the physics of the chapter, a familiar problem from high school and introductory college classes, meant to orient and refresh the reader. The mathematical lesson is about series solutions (the method of Frobenius) for ordinary differential equations (ODEs), and the definition of trigonometric special functions in terms of the ODEs that they solve. This is the chapter that reviews complex numbers and the basic properties of ODEs and their solutions (superposition, continuity, separation of variables).

Chapter 2 **Damped Harmonic Oscillator**: Here, we add damping to the harmonic oscillator, and explore the role of the resulting new timescale in the solutions to the equations of motion. Specifically, the ratio of damping to oscillatory timescale can be used to identify very different regimes of motion: under-, critically-, and over-damped. Then driving forces are added, we consider the effect those have on the different flavors of forcing already in place. The main physical example (beyond springs attached to masses in dashpots) is electrical, sinusoidally driven resistor, inductor, capacitor (RLC) circuits provide a nice, experimentally accessible test case. On the mathematical side, the chapter serves as a thinly veiled introduction to Fourier series and the Fourier transform.

Chapter 3 **Coupled Oscillators**: We turn next to the case of additional masses. In one dimension, we can attach masses by springs to achieve collective motions that occur at a single frequency, the normal modes. Building general solutions, using superposition, from this "basis" of solutions is physically relevant and requires a relatively formal treatment of linear algebra, the mathematical topic of the chapter.

Chapter 4 **The Wave Equation**: Taking the continuum limit of the chains of masses from the previous chapter, we arrive at the wave equation, the physical subject of this chapter. The connection to approximate string motion is an additional motivation. Viewed as a manifestation of a conservation law, the wave equation can be extended to other conservative, but nonlinear cases, like traffic flow. Mathematically, we are interested in turning partial differential equations

(PDEs) into ODEs, making contact with some familiar examples. Making PDEs into ODEs occurs in a couple of ways – the method of characteristics, and additive/multiplicative separation of variables are the primary tools.

Chapter 5 **Integration**: With many physical applications already on the table, in this chapter, we return to some of the simplified ones and recomplexify them. These problems require more sophisticated, and incomplete, solutions. Instead of finding the position of the bob for the simple pendulum, we find the period of motion for the "real" pendulum. Instead of the classical harmonic oscillator, with its familiar solution, we study the period of the relativistic harmonic oscillator, and find that in the high-energy limit, a mass attached to a spring behaves very differently from its nonrelativistic counterpart.

The eighth chapter, Numerical Methods, is used as a six-week "lab" component, one section each week. The chapter is relatively self-contained, and consists of numerical methods that complement the analytic solutions found in the rest of the book. There are methods for solving ODE problems (both in initial and boundary value form) approximating integrals, and finding roots. There is also a discussion of the eigenvalue problem in the context of approximate solutions in quantum mechanics and a section on the discrete Fourier transform.

There are two additional chapters that provide content when the book is used in an upper level setting, for third- or fourth-year students. In the sixth chapter, Waves in Three Dimensions, we explore the wave equation and its solutions in three dimensions. The chapter's mathematical focus is on vector calculus, enough to understand and appreciate the harmonic functions that make up the static solutions to the wave equation. Finally, the seventh chapter, Other Wave Equations, extends the discussion of waves beyond the longitudinal oscillations with which we began. Here, we look at the wave equation as it arises in electricity and magnetism (the three-dimensional context is set up in the previous chapter), in Euler's equation and its shallow water approximation, in "realistic" (extensible) strings, and in the quantum mechanical setting, culminating in a quantum mechanical treatment of the book's defining problem, the harmonic oscillator.

There are two appendices to provide review. The first reviews the basic strategy of ODE solving in a step-by-step way – what guesses to try, and when, with references to the motivating solutions in the text. The second appendix is a review of basic vector calculus expressions, like the gradient, divergence, curl, and Laplacian, in cylindrical, spherical, and more general coordinate systems.

My hope is that this book provides a focused introduction to many of the mathematical methods used in theoretical physics, and that the vehicles used to present the material are clear and compelling. I have kept the book as short as possible, yet tried to cover a variety of different tools and techniques. That coverage is necessarily incomplete, and for students going on in physics, a copy of one of the larger [1, 3, 15], and more sophisticated [2, 4, 17] mathematical methods texts will eventually be a welcome necessity, with this book providing some motivating guidance. (I encourage students to have one of these texts on hand as they read, so that when a topic like spherical Bessel functions comes up, they can look at the relevant section for additional information.)

Mary Boas has a wonderful "To the Student" section at the start of [3], an individual call to action that cannot be improved, so I will quote a portion of it:

> To use mathematics effectively in applications, you need not just knowledge, but *skill*. Skill can be obtained only through practice. You can obtain a certain superficial *knowledge* of mathematics by listening to lectures, but you cannot obtain *skill* this way. . . . The only way to develop the skill necessary to use this material in your later courses is to practice by solving many problems. Always study with pencil and paper at hand. Don't just read through a solved problem – try to do it yourself!

Since I was an undergraduate, I have always followed and benefited from this advice, and so, have included a large number of problems in this text.

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My own research background has informed some of the topics and presentation in this book, and that background has been influenced by many talented physicists and physics teachers – thanks to my mentors from undergraduate to postdoctoral, Professors Nicholas Wheeler, Stanley Deser, Sebastian Doniach, and Scott Hughes.

Finally, David Griffiths has, throughout my career been an honest sounding board, a source of clarity and wisdom. He has helped me both think about and present physics far better than I could on my own. I thank him for sharing his insights on this material and my presentation of it.