1 Harmonic Oscillator

The motivating problem we consider in this chapter is Newton's second law applied to a spring with spring constant k and equilibrium spacing a as shown in Figure 1.1.

The nonrelativistic equation of motion reads

$$m\ddot{x}(t) = -k(x(t) - a) \tag{1.1}$$

and we must specify initial (or boundary) conditions. The point of Newton's second law is the determination of the trajectory of the particle, $x(t)$ in this one-dimensional setting. The initial conditions render the solution unique. As a second-order differential equation (ODE), we expect the general solution to have two constants. Then we need two pieces of information beyond the equation itself to set those constants, and initial or boundary conditions can be used.

1.1 Solution Review

To proceed, we can define $k/m \equiv \omega^2$, so that our equation of motion becomes

$$\ddot{x}(t) = -\omega^2(x(t) - a)\,, \tag{1.2}$$

and finally, we let $y(t) \equiv x(t) - a$ in order to remove reference to a and allow us to identify the solution to this familiar ODE:

$$\ddot{y}(t) = -\omega^2 y(t) \longrightarrow y(t) = A\cos(\omega t) + B\sin(\omega t). \tag{1.3}$$

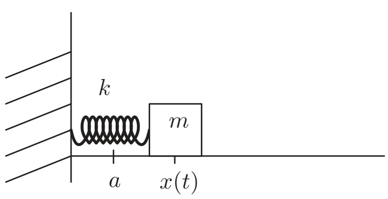

Fig. 1.1 A mass m is attached to a spring with spring constant k and equilibrium spacing a. It moves without friction under the influence of a force $F = -k(x(t) - a)$. We want to find the location of the mass at time t, $x(t)$, by solving Newton's second law.

The constants A and B have no *a priori* physical meaning, they are just the constants we get from a second-order ODE like (1.3).[1] The solution for $x(t)$ is

$$x(t) = y(t) + a = A\cos(\omega t) + B\sin(\omega t) + a. \tag{1.4}$$

Suppose we take the initial value form of the problem. We see the particle at x_0 at time $t = 0$, moving with velocity v_0. This allows us to algebraically solve for A and B:

$$\begin{aligned} x(0) &= A + a = x_0 \longrightarrow A = x_0 - a \\ \dot{x}(0) &= B\omega = v_0 \longrightarrow B = \frac{v_0}{\omega}. \end{aligned} \tag{1.5}$$

When we combine an ODE (like Newton's second law) with constants (the initial position and velocity), we have a well-posed *problem* and a unique solution. Putting it all together, the problem is

$$m\ddot{x}(t) = -k(x(t) - a) \qquad x(0) = x_0 \qquad \dot{x}(0) = v_0 \tag{1.6}$$

with solution

$$x(t) = (x_0 - a)\cos\left(\sqrt{\frac{k}{m}}t\right) + v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right) + a. \tag{1.7}$$

There are many physical observations and definitions associated with this solution. Suppose that we start the mass from rest, $v_0 = 0$, with an initial extension x_0, and we set the zero of the x axis at the equilibrium spacing a. Then the solution from (1.7) simplifies to

$$x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}t\right). \tag{1.8}$$

We call x_0 the "amplitude," the maximum displacement from equilibrium. The "period" of the motion is defined to be the time it takes for the mass to return to its starting point. In this case, we start at $t = 0$, and want to know when the "cosand" (argument of cosine) returns to $2\pi \sim 0$. That is, the period T is defined to be the first time at which

$$x(T) = x_0\cos\left(\sqrt{\frac{k}{m}}T\right) = x_0 \longrightarrow \sqrt{\frac{k}{m}}T = 2\pi \longrightarrow T = 2\pi\sqrt{\frac{m}{k}}. \tag{1.9}$$

This period is, famously, independent of the initial extension.[2] That makes some sense, physically – the larger the initial extension, the faster the maximum speed of the mass is, so that even though it has to travel a longer distance, it does so at a greater speed. Somehow, magically, the two effects cancel in this special case.

We can also define the "frequency" of the oscillatory motion, that is just the inverse of the period, $f \equiv 1/T$. For the mass on a spring motion,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \tag{1.10}$$

[1] These constants are called "constants of integration" and are reminiscent of the constants that appear when you can actually integrate the equation of motion twice. That happens when, for example, a force depends only on time. Then you can literally integrate $\ddot{x}(t) = F(t)/m$ twice to find $x(t)$. There will be two constants of integration that show up in that process.

[2] Don't believe it! See Section 5.5.

and we define the "angular frequency" of the oscillatory motion to be

$$\omega \equiv 2\pi f = \sqrt{\frac{k}{m}}, \tag{1.11}$$

where ω is the letter commonly used, and we have taken advantage of that in writing (1.2).

Problem 1.1.1 What is the solution to Newton's second law (1.1) with *boundary* values given: $x(0) = x_0$ and $x(t^*) = x_*$ (t^* refers to some specific time at which we are given the position, x_*)?

Problem 1.1.2 What happens to the solution in the previous problem if $\omega t^* = n\pi$ for integer n?

Problem 1.1.3 Solve $m\ddot{x}(t) = F_0$ for constant force F_0 subject to the boundary conditions: $x(0) = x_0$, $x(t^*) = x_*$ with x_0 and x_* given. Solve the same problem for a "mixed" set of conditions: $x(0) = x_0$ and $\dot{x}(t^*) = v_*$ with x_0 and v_* given.

Problem 1.1.4 For the oscillatory function $x(t) = x_0 \cos(\omega t + \phi)$ with constant $\phi \in [0, 2\pi)$ (the "phase"), find the amplitude and period, and sketch one full cycle of this function.

Problem 1.1.5 Suppose Newton's second law read: $\alpha \dddot{x}(t) = F(x(t), t)$ for force F. What are the units of α in this case? Solve the modified Newton's second law if the force is a constant F_0 with initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. What is the problem with this solution? (is it, for example, unique?).

Problem 1.1.6 Solve $m\ddot{x}(t) = F_0 \cos(\omega t)$ (F_0 is a constant with Newtons as it unit, ω is a constant with unit of inverse seconds) for $x(t)$ given $x(0) = x_0$ and $\dot{x}(0) = v_0$.

1.2 Taylor Expansion

To appreciate the role of the harmonic oscillator problem in physics, we need to review the idea of expanding a function $f(x)$ about a particular value x_0 and apply it to minima of a potential energy. We'll start in this section with the former, called "Taylor expansion." The idea is to estimate $f(x_0 + \Delta x)$ for small Δx given the value of the function and its derivatives at x_0. Our first guess is that the function is unchanged at $x_0 + \Delta x$,

$$f(x_0 + \Delta x) \approx f(x_0). \tag{1.12}$$

That's a fine approximation, but can we improve upon it? Sure: if we knew $f(x_0)$ and

$$f'(x_0) \equiv \left. \frac{df(x)}{dx} \right|_{x=x_0}, \tag{1.13}$$

then we could add in a correction associated with the slope of the line tangent to $f(x_0)$ at x_0:

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x. \tag{1.14}$$

The picture of this approximation, with the initial estimate and the linear refinement is shown in Figure 1.2.

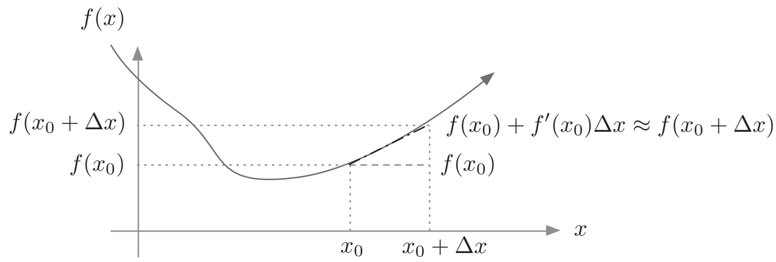

Fig. 1.2 Given a function $f(x)$, if we know the value of the function and its derivative at x_0, then we can estimate the value at a nearby point $x_0 + \Delta x$ for Δx small.

The process continues, we can take the quadratic correction, $f''(x_0)\Delta x^2$, and use it to refine further,

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x + \frac{1}{2} f''(x_0)\Delta x^2. \tag{1.15}$$

You can keep going to any desired accuracy, with equality restored when an infinite number of terms are kept,

$$f(x_0 + \Delta x) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{d^j}{dx^j} f(x) \right) \bigg|_{x=x_0} \Delta x^j, \tag{1.16}$$

although we rarely need much past the first three terms. Note that the first term that you drop gives an estimate of the error you are making in the approximation. For example, if you took only the first two terms in (1.15), you would know that the leading source of error goes like $f''(x_0)\Delta x^2$.

The hardest part of applying Taylor expansion lies in correctly identifying the function $f(x)$, the point of interest, x_0, and the "small" correction Δx. As an example, suppose we wanted to evaluate $\sqrt{102}$, then we have $f(x) = \sqrt{x}$ with $x_0 = 100$ and $\Delta x = 2$. Why pick $x_0 = 100$ instead of 101? Because it is easy to compute $f(100) = \sqrt{100} = 10$, while for $\sqrt{101}$ we have the same basic problem we started out with (i.e. I don't know the value of $\sqrt{101}$ any more than I know $\sqrt{102}$). Using (1.15) gives the estimate

$$\begin{aligned} f(100 + 2) &\approx f(100) + f'(100) \times 2 + \frac{1}{2} f''(100) \times 4 \\ &= \sqrt{100} + \frac{1}{2\sqrt{100}} \times 2 - \frac{1}{2}\frac{1}{4(100)^{3/2}} \times 4 \\ &= 10 + \frac{1}{10} - \frac{1}{2000} = 10.0995 \end{aligned} \tag{1.17}$$

while the "actual" value is $\sqrt{102} \approx 10.099505$.

Problem 1.2.1 Evaluate $\sin(\Delta x)$ and $\cos(\Delta x)$ for the values of Δx given in the following table using a calculator. Then use the Taylor expansions of these functions (to second order, from (1.15)) to approximate their value at those same Δx (assuming Δx is small, an assumption that is violated by some of the values so you can see the error in the approximations). Write out your results to four places after the decimal.

Δx	$\sin(\Delta x)$	$\cos(\Delta x)$	$\sin(\Delta x)$ approx.	$\cos(\Delta x)$ approx.
.1				
.2				
.4				
.8				

Problem 1.2.2 Use Taylor expansion to find the approximate value of the function $f(x) = (a+x)^n$ for constants a and n with $x_0 = 0$, i.e. what is $f(\Delta x)$ for Δx small (take only the "leading-order" approximation, in which you just write out the Taylor expansion through the Δx term as in (1.14))? Using your result, give the Taylor expansion approximation near zero (again, you'll write expressions approximating $f(\Delta x)$) for:

$$f(x) = \sqrt{1+x} \approx$$

$$f(x) = \frac{1}{\sqrt{1+x}} \approx$$

$$f(x) = \frac{1}{(1+x)^2} \approx$$

Problem 1.2.3 Estimate the value of 1/121 using Taylor expansion (to first order in the small parameter) and compare with the value you get from a calculator. Hint: $121 = (10+1)^2$.

Problem 1.2.4 For the function $f(\theta) = (1+\cos\theta)^{-1}$, estimate $f(\pi/2 + \Delta\theta)$ for $\Delta\theta \ll 1$ using (1.15). Try it again by Taylor expanding the $\cos\theta$ function first, then expanding in the inverse polynomial, a two-step process that should yield the same result (up to errors that we have ignored in both cases).

1.3 Conservative Forces

The spring force starts off life as rusty bits of metal providing a roughly linear restoring force. But the model's utility in physics has little to do with the coiled metal itself. Instead, the "harmonic" oscillator behavior is really the dominant response of a particle moving near the equilibrium of a potential energy function. To review, a conservative force F comes from the derivative of a potential energy U via:

$$F(x) = -\frac{dU(x)}{dx}. \tag{1.18}$$

If we have a potential energy $U(x)$ (from whatever physical configuration), then a point of equilibrium is defined to be one for which the force vanishes. For x_e a point of equilibrium,

$$F(x_e) = 0 = -\left.\frac{dU(x)}{dx}\right|_{x=x_e} \equiv -U'(x_e). \tag{1.19}$$

Now if we expand the potential energy function $U(x)$ about the point x_e using Taylor expansion:

$$U(x) = U(\underbrace{(x - x_e)}_{\equiv \Delta x} + x_e) = U(x_e) + \Delta x U'(x_e) + \frac{1}{2}\Delta x^2 U''(x_e) + \cdots, \tag{1.20}$$

then the first term is just a constant, and that will not contribute to the force in the vicinity of x_e (since we take a derivative with respect to x sitting inside Δx to get the force). The second term vanishes by the assumption that x_e is a point of equilibrium, and the first term that informs the dynamics of a particle moving in the vicinity of x_e is the third term $\sim (1/2)U''(x_e)(x - x_e)^2$, leading to a force, near x_e:

$$F(x) = -\frac{dU(x)}{dx} \approx -U''(x_e)(x - x_e) + \cdots \tag{1.21}$$

The effective force in the vicinity of the equilibrium is just a linear restoring force with "spring constant" $k \sim U''(x_e)$ (assuming $U''(x_e) > 0$ so that the equilibrium represents a local *minimum*) and equilibrium spacing x_e. A picture of a local minimum in the potential energy and the associated force is shown in Figure 1.3. Near x_e, the potential is approximately quadratic, and the force is a linear restoring force of the sort we have been studying. There is also an equilibrium point at the maximum of $U(x)$ in that picture, but the associated force tends to drive motion *away* from this second equilibrium. We call such locations points of "unstable equilibrium," even a small perturbation from the equilibrium location drives masses away.

As an example, suppose we have somehow managed to set up a potential energy of the form $U(x) = U_0 \cos(2\pi x/\ell)$ for a length ℓ and constant $U_0 > 0$. What is the period of motion for a particle that starts out "near" $x_e = \ell/2$? In this case, the equilibrium position

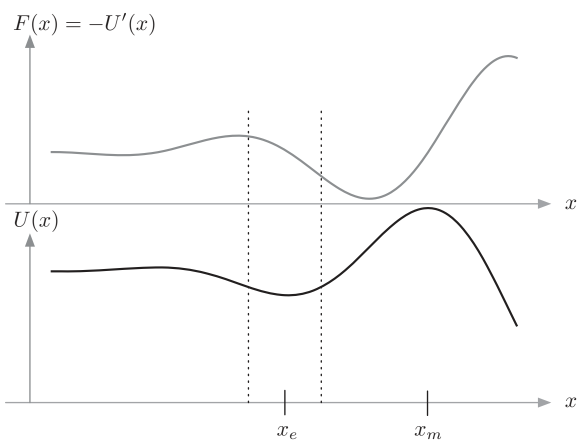

Fig. 1.3 A potential energy $U(x)$ (lower plot) with associated force $F(x) = -U'(x)$ (upper). A minimum in the potential has an approximately linear restoring force associated with it. We can approximate the force, in the vicinity of x_e (bracketed with dotted lines) by $F(x) \approx -U''(x_e)(x - x_e)$. The maximum at x_m is also a point of equilibrium, but this one is "unstable," if a particle starts near x_m it tends to get driven away from x_m (the slope of the force is positive).

is at $x_e = \ell/2$, with $U'(x_e) = 0$ as required. The second derivative sets the effective spring constant, $k = U''(x_e) = (2\pi/\ell)^2 U_0$, and then the period and frequency of the resulting oscillatory motion come from (1.9) and (1.10):

$$T = 2\pi\sqrt{\frac{m}{U''(x_e)}} = \sqrt{\frac{m\ell^2}{U_0}} \qquad f = \sqrt{\frac{U_0}{m\ell^2}}. \tag{1.22}$$

1.3.1 Conservation of Energy

It is worth reviewing the utility of conservation of energy. In particular, while we have a complete solution to our problem, in (1.4), it is not always possible to find such a complete solution. In those cases, we retreat to a "partial" solution, where we can still make quantitative predictions (and hence physical progress), but we might not have the "whole" story.

Let's go back to Newton's second law, this time for an arbitrary potential energy $U(x)$, but with initial values specified. Think of the ODE piece of the "problem":

$$m\ddot{x}(t) = -\frac{dU(x)}{dx}. \tag{1.23}$$

Now, the only ODEs that I can solve in closed form are ones in which both sides are a total derivative, in this case, a total *time*-derivative. Then to integrate, you just remove the $\frac{d}{dt}$ from both sides, and add in a constant – that process returns a function for $\dot{x}(t)$. Then, when possible, you integrate again to get $x(t)$ (picking up another constant). This direct approach will be considered in Section 5.1.

Looking at the left side of (1.23), it is clear that we have a total time derivative: $m\ddot{x}(t) = \frac{d}{dt}(m\dot{x}(t))$, but what about the right-hand side? Is there a function $W(x)$ such that

$$\frac{dW(x(t))}{dt} = -\frac{dU(x)}{dx}? \tag{1.24}$$

The answer is *no*. The reason is clear: If we had a function evaluated at $x(t)$, $W(x(t))$, then the total time derivative of W would look like

$$\frac{dW(x(t))}{dt} = \frac{dW(x)}{dx}\frac{dx(t)}{dt} = \frac{dW(x)}{dx}\dot{x}(t) \tag{1.25}$$

and there is no $\dot{x}(t)$ that appears on the right in (1.23). The fix is easy, just put an $\dot{x}(t)$ on the right-hand side of (1.23), which requires putting one on the left-hand side as well. Then Newton's second law looks like

$$m\dot{x}(t)\ddot{x}(t) = -\frac{dU(x)}{dx}\dot{x}(t). \tag{1.26}$$

The situation on the right is now very good, since we can write the right-hand side as a total time derivative:

$$-\frac{dU(x(t))}{dt} = -\frac{dU(x)}{dx}\dot{x}(t). \tag{1.27}$$

There is potential trouble on the *left*-hand side of Newton's second law, though. Can $\dot{x}(t)\ddot{x}(t)$ be written as a total time derivative? Yes, note that

$$\frac{d}{dt}\left(\dot{x}(t)^2\right) = 2\dot{x}(t)\ddot{x}(t). \tag{1.28}$$

Just multiplying Newton's second law by $\dot{x}(t)$ on both sides has given us the integrable equation

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}(t)^2\right) = -\frac{dU(x(t))}{dt} \longrightarrow \frac{1}{2}m\dot{x}(t)^2 = -U(x(t)) + E \tag{1.29}$$

where E is the constant of integration. We could re-write (1.29) as

$$\frac{1}{2}m\dot{x}(t)^2 + U(x(t)) = E. \tag{1.30}$$

This represents an interesting situation – the combination of the time-dependent terms on the left yields a time *in*-dependent term on the right. Thinking about units (or dimensions, if that's what you are into) we have, in the first term of (1.30) a "kinetic" energy (energy because of the units, kinetic because the term is associated with movement through its dependence on $\dot{x}(t)$) and a "potential" energy (again from units, and the dependence on, this time, position). The sum is a constant of the motion of the particle, E, the "total" energy of the system. That is the statement of energy conservation expressed by (1.30). Because E is a constant, we can set its value from the provided initial conditions: $x(0) = x_0, \dot{x}(0) = v_0$,

$$\frac{1}{2}mv_0^2 + U(x_0) = E. \tag{1.31}$$

The integration of Newton's second law, in the presence of a "conservative" force, given by (1.30) is notable for its predictive ability – if you tell me where the particle is, its location at time t, $x(t)$, I can tell you how fast it is moving. Using the constant value of E set by the initial conditions for the motion (1.31), we can write (1.30) as

$$\frac{1}{2}m\dot{x}(t)^2 + U(x(t)) = \frac{1}{2}mv_0^2 + U(x_0), \tag{1.32}$$

and then

$$\dot{x}(t) = \pm\left[v_0^2 + \frac{2}{m}\left(U(x_0) - U(x(t))\right)\right]^{1/2} \tag{1.33}$$

gives the speed (taking the positive root), at time t, of the particle at location $x(t)$.

1.3.2 Harmonic Oscillator

The harmonic oscillator potential energy is just the quadratic $U(x) = 1/2k(x-a)^2$ for equilibrium location a. In this case, the quadratic expansion of $U(x)$ about the equilibrium consists of just the one term. We know what happens here: a particle oscillates about the equilibrium value with frequency governed by $\sqrt{k/m}$. If you think of the graph of the potential energy, we have a convex curve, and if you draw a line representing energy E as in Figure 1.4, you can tell the "story of the motion": where the value of E intersects the

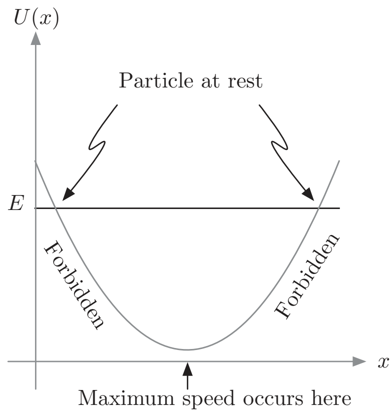

Fig. 1.4 A particle with energy E moving in a quadratic potential well. The particle is at rest where E intersects the potential energy function, and achieves its maximum speed where the difference between E and $U(x)$ is largest.

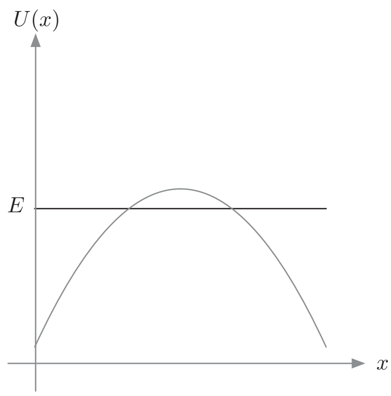

Fig. 1.5 A particle with energy E moving in a quadratic potential with a maximum. The particle cannot exist "underneath" the potential energy curve.

potential energy, the particle must be at rest (all the energy is potential, none is kinetic), the point at which the difference between E and $U(x)$ is largest represents the position at which the maximum speed of the particle occurs. Locations where $U(x) > E$ are impossible to achieve physically, since the kinetic energy would have to be negative.

What if the potential energy had the form $U(x) = -1/2k(x-a)^2$, with a concave graph? This time, a particle tends to move *away* from the equilibrium position, without returning to it. Thinking of motion at a fixed E, as in Figure 1.5, the particle speeds up as it gets further away from the equilibrium location. This is an example of an "unstable" equilibrium, particles that start near a are driven away from it. For the usual harmonic potential, with its $+$ sign, the equilibrium point is "stable," if you start near equilibrium, you remain near it. In a more general potential energy landscape, the sign of the second derivative of the potential energy function, evaluated at a point of equilibrium, determines whether the equilibrium is stable (positive) or unstable (negative).

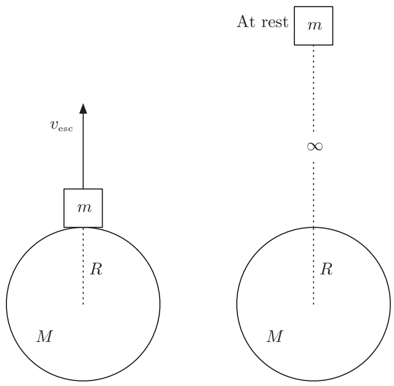

Fig. 1.6 A mass m leaves the surface of the Earth with speed v_{esc} and comes to rest infinitely far away.

1.3.3 Escape Speed

Conservation of energy can be used to make quantitative predictions even when the full position of an object as a function of time is not known. As an example, consider a spherical mass M of radius R. A test particle[3] of mass m at a distance $r \geq R$ from the center of the sphere experiences a force with magnitude

$$F = \frac{GMm}{r^2} \tag{1.34}$$

where G is the gravitational constant. This force comes from a potential energy $U(r) = -GMm/r$. Suppose the test mass starts from the surface of the sphere with a speed v_{esc}. We want to know the minimum value for v_{esc} that allows the test mass to escape the gravitational pull of the spherical body. That special speed is called the "escape speed." Think of the spherical body as the earth, and the test mass is a ball that you throw up into the air. The ball goes up and comes back down. If you throw the ball up in the air a little faster, it takes longer to come down. The escape speed is the (minimum) speed at which you must throw the ball up so that it *never* comes back down.

Formally, we want the test particle to reach $r \to \infty$ where it will be at rest as shown on the right in Figure 1.6.[4] From (1.30), if we take "$x(t) = r \to \infty$" with "$\dot{x}(t) = \dot{r} \to 0$," and use the potential energy associated with gravity, we have $E = 0$. Going back to the initial values, which must of course have the same energy,

$$\frac{1}{2} m v_{esc}^2 + U(R) = E = 0, \tag{1.35}$$

and then the escape speed can be isolated algebraically

$$v_{sc} = \sqrt{-\frac{2}{m} U(R)} = \sqrt{\frac{2GM}{R}}. \tag{1.36}$$

[3] "Test particle" is a technical term that means "a particle that feels the effect of a force without contributing to it." When we want to probe the gravitational force associated with some external body, we often imagine a test particle's response to that force. The same idea shows up in electricity and magnetism.

[4] That's the "minimum" part of the requirement. You could have the test particle rocketing around at spatial infinity, but that excess speed is overkill.