The Bellman Function Technique in Harmonic Analysis

The Bellman function, a powerful tool originating in control theory, can be used successfully in a large class of difficult harmonic analysis problems and has produced some notable results over the last 30 years. This book by two leading experts is the first devoted to the Bellman function method and its applications to various topics in probability and harmonic analysis. Beginning with basic concepts, the theory is introduced step-by-step starting with many examples of gradually increasing sophistication, culminating with Calderón–Zygmund operators and endpoint estimates. All necessary techniques are explained in generality, making this book accessible to readers without specialized training in nonlinear PDEs or stochastic optimal control. Graduate students and researchers in harmonic analysis, PDEs, functional analysis, and probability will find this to be an incisive reference, and can use it as the basis of a graduate course.

Vasily Vasyunin is Leading Researcher at the St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences and Professor at the Saint Petersburg State University. His research interests include linear and complex analysis, operator models, and harmonic analysis. Vasyunin has taught at universities in Europe and the United States. He has authored or coauthored over 60 articles.

Alexander Volberg is Distinguished Professor of Mathematics at Michigan State University. He was the recipient of the Onsager Medal as well as the Salem Prize, awarded to a young researcher in the field of analysis. Along with teaching at institutions in Paris and Edinburgh, Volberg also served as a Humboldt senior researcher, Clay senior researcher, and a Simons fellow. He has coauthored 179 papers, and is the author of *Calderón–Zygmund Capacities and Operators on Non-Homogenous Spaces*.

Cambridge University Press
978-1-108-48689-7 — The Bellman Function Technique in Harmonic Analysis
Vasily Vasyunin , Alexander Volberg
Frontmatter
More Information

The Bellman Function Technique in Harmonic Analysis

VASILY VASYUNIN
Russian Academy of Sciences

ALEXANDER VOLBERG
Michigan State University

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108486897
DOI: 10.1017/9781108764469

First published 2020

Printed in the United Kingdom by TJ International, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Vasyunin, Vasily I., 1948– author. | Volberg, Alexander, 1956– author.
Title: The Bellman function technique in harmonic analysis /
Vasily Vasyunin, Alexander Volberg.
Description: Cambridge ; New York, NY : Cambridge University Press, 2020. |
Includes bibliographical references and index.
Identifiers: LCCN 2019042603 (print) | LCCN 2019042604 (ebook) |
ISBN 9781108486897 (hardback) | ISBN 9781108764469 (epub)
Subjects: LCSH: Harmonic analysis. | Functional analysis. | Control theory.
Classification: LCC QA403 .V37 2020 (print) | LCC QA403 (ebook) |
DDC 515/.2433–dc23
LC record available at https://lccn.loc.gov/2019042603
LC ebook record available at https://lccn.loc.gov/2019042604

ISBN 978-1-108-48689-7 Hardback

To our parents

Contents

Introduction

I.1 Preface

The subject of this book is the use of the Bellman function technique in harmonic analysis. The Bellman function, in principle, is the creature of another area of mathematics: control theory. We wish to show that it can be used very successfully in a big class of harmonic analysis problems. In the last 25–30 years some outstanding problems in harmonic analysis were solved by this approach. Later, 10–15 years after, another solution has been found by more classical methods involving some highly nontrivial stopping time argument.

This is what happened with the A_2 conjecture and then very recently with the A_1 conjecture concerning weighted estimates of singular integrals. Some other problems solved by the Bellman function method still await their "de-Bellmanisation." Among such problems, we can list the celebrated solution by Burkholder of Pełczyński's problem about Haar basis, the best L^p estimates of the Ahlfors–Beurling operator, and many matrix weight estimates.

One of the main technical advantages of the Bellman function technique is that it does not require the invention of any sophisticated stopping time argument of the kind that is so pervasive in modern harmonic analysis. We can express this feature by saying that the Bellman function knows how to stop the time correctly, but it does not show us its secret.

The purpose of this book is to present a wide range of problems in harmonic analysis having the same underlying structure that allows us to look at them as problems of stochastic optimal control and, consequently, to treat them by the methods originated from this part of control theory.

We intend to show that a certain class of harmonic analysis problems can be reduced (often without any loss of information) to solving a special partial differential equation called the Bellman equation of a problem. For that purpose, we first cast a corresponding harmonic analysis problem as a stochastic optimization problem.

A quintessentially typical problem of harmonic analysis is to find (or estimate) the norm of this or that (singular) operator in function space L^p. If we think about the operator as a black box, we should think about the unit ball of L^p as its input. The unit ball of L^p is not compact in norm topology, so there is a priori no extremizer, and moreover, it seems to be very difficult to "list" all functions in the unit ball and "try" them as black box inputs one by one.

The stochastic point of view helps here, because we can think about input as a stochastic process stopped at a certain time. This point of view gives a very nice and powerful way to list all inputs as solutions of simple stochastic differential equations with some unknown stochastic control. Then the norm of the operator becomes a functional on solutions of stochastic differential equations that we need to optimize by choosing optimal control.

The technique of doing that is to consider the Bellman function of this control problem and to write the Hamilton–Jacobi–Bellman equation whose solution the Bellman function is supposed to be.

This book can be used as the basis of a graduate course, and it can also serve as a reference on many (but not all) applications of the Bellman function technique in harmonic analysis.

A certain number of very important results obtained with the use of the Bellman function technique stayed outside of the scope of this book. For example, these are the twisted paraproducts results of V. Kovac [**91, 92**], and, in general, the applications of the Bellman function to multilinear and nonlinear harmonic analysis. In the last category one finds the works of C. Muscalu, T. Tao, and C. Thiele [**119**] concerning nonlinear analogs of the Hausdorff–Young inequality that relates the norm of a function and the norm of its nonlinear Fourier transform. This book does not present the recent results of O. Dragicevic and A. Carbonaro [**33, 34**], where the authors study universal multiplier theorems in the setting of symmetric contraction semigroups. In particular, the authors solved a long-standing problem of finding the optimal sector, where generators of symmetric contraction semigroups always admit a H^∞-type holomorphic functional calculus on L^p. This is done by a subtle application of the Bellman function technique on a flow that is given by the semigroup. Numerous multiplier theorems are improved due to this result, and new results on pointwise convergence related to a symmetric contraction semigroup on a closed sector are obtained.

The corresponding papers can be found in the References, and the reader is encouraged to study these beautiful applications of the Bellman function ideology.

I.2 Acknowledgments

Our foremost acknowledgments are to our teachers M. S. Birman and N. K. Nikolski. We also learned most of what we know from our collaborators, colleagues, and our students: Fedor Nazarov, Sergei Treil, Igor Verbitskiy, Dmitriy Stolyarov, Pavel Zatitskiy, Stefanie Petermichl, Paata Ivanisvili, Alexander Reznikov, Oliver Dragicevic, Irina Holmes, Leonid Slavin, and many others. The ideas we caught while working and/or talking with them are incorporated in this book. Special thanks are due to Marina Kuznetsov who corrected our most horrible blunders in English.

We are very grateful to Oberwolfach Mathematics Research Institute, to Centre International de Rencontres Mathématiques in Luminy, and to Department of Mathematics of Michigan State University for excellent working conditions and invitations for the two of us together to stay during 2014–2017. We are also grateful to NSF support that allowed us to work together and meet regularly.

Finally, we would like to thank our friends and relatives who encouraged us during writing of this book, especially our wives, Nina Vasyunina and Olga Volberg.

I.3 The Short History of the Bellman Function

The Bellman equation and the Euler–Lagrange equation both deal with extremal values of functionals. Given a functional, the Euler–Lagrange approach gives us a differential equation that rules the behavior of extremizers of a functional in question. The Bellman equation has quite a different nature. At first glance, it does not give any information on the extremizers. Moreover, typically, there will be no extremal function. Only a sequence of almost extremizers typically exists, which is yet another difference with applications of the Euler–Lagrange equation.

In the Bellman paradigm, the (system of) differential equation(s) is given to us, and we do not need to find it as in the Euler–Lagrange approach. However, the given differential equation (or a system of equations) also has an unknown functional parameter called control. We have to find the best control in the sense that this control will optimize a functional applied to the solution of (an) already given (system of) differential equation(s).

So the idea of Bellman is amazingly striking and incredibly simple simultaneously. Let us denote by $\mathbf{B}(x)$ the extremal value of the functional that we want to optimize on solutions of a given system of differential equations with initial value at x at initial time 0. Using the fact that at time Δt, we "know" where the solution is, and that having started at $x(\Delta t)$ this solution is also extremal (if the control is chosen correctly), one can deduce a partial differential equation on $\mathbf{B}$.

This equation is called the Hamilton–Jacobi equation. Due to the functional in need of optimization, and depending on the system of differential equations that is given to us, the Hamilton–Jacobi equation varies, but it stays in a certain class of first-order nonlinear (usually) PDEs.

However, if the system of differential equations mentioned previously is not the usual system, but is a system of stochastic differential equations, and we are asked to optimize not just a functional of solutions of this system, but the expectation (the average) of this functional, then the scheme mentioned previously can be applied as well. The resulting PDE is often called the Hamilton–Jacobi–Bellman equation, and the presence of stochasticity makes it a second-order nonlinear (usually) PDE. It belongs to the class of equations called degenerate elliptic equations, see, e. g., [**120**].

Previously we presented a very short exposition of the Bellman function of stochastic optimal control. In this branch of mathematics, it is also often called the value function. The reader who wants to acquire good knowledge in this area is advised to read [**95**].

But the goal of the book is to show the deep (and almost perfect) analogy between the Bellman function technique in stochastic optimal control and the Bellman function technique in harmonic analysis, which is the branch of analysis dealing with the estimates of singular integrals.

It was arguably in [**130**], and especially in [**131, 193**], where this parallelism between a wide class of harmonic analysis problems and the stochastic optimal control got recognized at face value.

The observation of this parallelism between two different branches of mathematics is sort of important for this book, but the ideas that now are generally recognized as the "Bellman function technique" in harmonic analysis have been around long before those papers.

Without any claim of completeness, we can list several articles and their ideas that now can be recognized as instances of application of the Bellman function technique (without ever mentioning stochastic control, the value function, or anything like that).

In the area of probability theory that deals with optimal problems for Brownian motion or for martingales, D. Burkholder [**22–31**], B. Davis [**50**], and Burkholder–Gundy [**32**] used what we call now the Bellman function as their main tool of finding the constants of best behavior of stopping times and of martingales with various restrictions.

But to the best of our knowledge, the first use of the idea underlying the Bellman function technique is due to A. Beurling, who found the exact function of uniform convexity for the space $L^p(0,1)$. Strangely enough, his work was not published: Beurling just made an oral report in Uppsala in 1945, and the exposition of his idea can be found in the paper of O. Hanner [**66**] of 1956. However, Beurling used certain magic guesses. These guesses were explained in the paper of Ivanisvili–Stolyarov–Zatitskiy [**82**], who showed that Beurling's function method is nothing other than perhaps the first occasion of the application of the Bellman function technique in harmonic analysis.

We think that chronologically the next case of using the Bellman function technique in harmonic analysis was again related to uniform convexity. But this case deals with the general theory of Banach spaces. In 1972, P. Enflo in [**59**] proved that the Banach space X is super-reflexive if and only if it can be given an equivalent norm that is uniformly convex. In fact, the "if" part was proved by R. C. James in [**86**]. Enflo proved the "only if" part, and the proof of Lemma 2 of [**59**] now reads as a typical Bellman function technique proof.

In 1975, G. Pisier [**154**] gave another proof of the James–Enflo result that the Banach space X is super-reflexive if and only if it has an equivalent uniformly convex norm. His proof used X-valued martingale interpretation of super-reflexivity. The uniformly convex norm on X was constructed in the second line of the proof of Theorem 3.1 of [**154**]. In fact, for a vector $x \in X$ this equivalent norm $|x|$ is defined as the infimum of a certain functional on X-valued martingales starting at x, and this is a quintessential Bellman function definition.

Let us briefly explain why we associate such an approach (also used in all the papers of Burkholder, Gundy, Davis mentioned previously) with the Bellman function technique described previously.

Roughly speaking, any martingale is a solution of a controlled stochastic differential equation (with continuous or discrete time), where martingale differences play the role of control that should be optimized to give a prescribed functional on martingale the "best" value. In the case explained in [**154**], the functional is given at the beginning of the proof of Theorem 3.1, and its optimal value is precisely $|x|$ – the equivalent norm of the initial vector x, where a martingale (the solution of a stochastic differential equation in our interpretation) has started.

The fact that $x \to |x|$ is uniformly convex is exactly the Hamilton–Jacobi–Bellman PDE, as the reader will conclude after reading this book. It sounds strange: Why should a certain inequality be called a partial differential equation?

It will be explained repeatedly that the Bellman PDE pertinent to a harmonic analysis problem is quite often, in fact, a certain second-order finite difference inequality.

It is difficult to find the optimal solutions of inequalities, so the reader will see in a case-by-case study how we account for this difficulty and how we remedy it.

In probability theory, there was an interest in understanding the relationship between the various norms of the stopping time T and the corresponding norms of $W(T)$, where W is the Brownian motion. The exposition of these results (and related martingale results) of B. Davis [**50**], G. Wang [**196**], and [**197**] can be found in Chapter 5.

A huge amount of work has been done in the papers by D. Burkholder [**21, 22**] and by Burkholder and Gundy [**32**]. These are all Bellman function technique papers. In particular, this method (without mentioning any stochastic optimal control or Hamilton–Jacobi equation) was used by Burkholder (in his seminal articles cited previously) to solve problems of A. Pełczyński, concerning sharp constants for unconditional Haar basis in L^p. We adapt Burkholder's solution to our language of the Bellman function technique, which is done in Section 1.8. This is one of those cases when it is easy to write the Bellman equation but difficult to solve it.

I.4 The Plan of the Book

In Chapter 1, we give nine precise Bellman functions corresponding to several typical harmonic analysis problems. As we already mentioned, Section 1.8 of this chapter is devoted to Burkholder's Bellman function. The John–Nirenberg inequality presents a very nice model for the application of the Bellman function technique, which the reader will find in Section 1.3. Then, in Section 1.5 we extend the method of the Bellman function to rather general functionals on the space BMO.

In Chapter 2, we first list elements of stochastic calculus and introduce the Bellman function of stochastic optimal control. Then in Section 2.6, we collect examples that show the perfect analogy between stochastic optimal control and a wide class of harmonic analysis problems. After that, we turn our attention to a class of problems from complex analysis that also can be adapted to the Bellman method. One of these problems is finding Pichorides constants, yet another question is concerned with the solution of Gohberg–Krupnik problem by B. Hollenbeck and I. Verbitsky [**69**]. Our main goal is to show that all harmonic analysis problems in this chapter can be interpreted as problems of stochastic optimal control. An important disclaimer should be made: the stochastic optimal control point of view helps us to write down a correct Bellman partial differential equation, but it does not, in any sense, help to solve it. And solving it can be a major difficulty.

Chapter 3 is devoted to sharp estimates of conformal martingales. We then use these results to consider one particular singular integral, the Ahlfors–Beurling transform. We give the best up-to-date estimates of this transform.

Chapter 4 demonstrates an interesting and unexpected feature of the Bellman function technique. Namely, it has been noticed that the Bellman function built for one problem can be used in another problem, sometimes not too close to the original problem. This allows us to use the Bellman functions for the weighted martingale transform to have the right estimates for much more complicated dyadic singular operators, the so-called dyadic shifts. Moreover, one need not know the precise form of the Bellman function, one should just know of its existence. This idea for the Ahlfors–Beurling transform was used by S. Petermichl and A. Volberg in [**152**]. In that chapter, we follow the ideas of S. Treil [**181**] with a slight modification.

It has been noticed repeatedly that the Bellman function technique can be used not only to prove the conjectural estimates of singular integrals, but also to disprove the estimates. This is, roughly speaking, the consequence of the fact that the language of Bellman functions is often exactly adequate and equivalent to harmonic analysis problems for which these functions are built. This observation helps to find sharp constants in several endpoint estimates for singular integrals. That point of view also brings counterexamples to several well-known conjectures. We devote Section 5.2 to such counterexamples. The rest of this chapter is devoted to using the Bellman function technique to find sharp estimates in several classical problems concerning the square function operator. Even though the sharp constants for this operator have been studied since 1975, there are still open questions and we discuss them in Chapter 5.

I.5 Notation

We conclude this Introduction by a short list of notation that will be used throughout the whole book.

The average of a summable function w over an interval I will be denoted by the symbol $\langle w\rangle_I$:

$$\langle w\rangle_I \stackrel{\text{def}}{=} \frac{1}{|I|}\int_I w(t)\,dt\,,$$

where $|I|$ stands for the Lebesgue measure of I.

We introduce the Haar system, normalized in L^∞:

$$H_I(t) = \begin{cases} -1 & \text{if } t \in I_-, \\ 1 & \text{if } t \in I_+, \end{cases} \tag{0.1}$$

and another one, normalized in L^2:

$$h_I(t) = \frac{1}{\sqrt{|I|}} H_I(t).$$

Then

$$|I|(\langle w\rangle_{I_+} - \langle w\rangle_{I_-}) = 2(w, H_I)$$

and

$$\sqrt{|I|}(\langle w\rangle_{I_+} - \langle w\rangle_{I_-}) = 2(w, h_I).$$

The characteristic function of a measurable set E is denoted by $\mathbf{1}_E$.

The symbol $\mathcal{D}$ stands for a dyadic lattice, and $\mathcal{D}_n$ stands for the grid of intervals (or cubes) of length (or side-length) 2^{-n}, $n \in \mathbb{Z}$. The σ-algebra generated by $\mathcal{D}_n$ is denoted by $\mathcal{F}_n$.

The symbol $\mathbb{E}_n$ stands for the expectation with respect to the σ-algebra $\mathcal{F}_n$. Then Δ_n stands for $\mathbb{E}_{n+1} - \mathbb{E}_n$, $\Delta_I \stackrel{\text{def}}{=} \mathbf{1}_I \Delta_n$ for $I \in \mathcal{D}_n$, and thus,

$$\Delta_n = \sum_{I \in \mathcal{D}_n} \Delta_I\,.$$

Bellman functions are usually denoted by $\mathbf{B}$, but in Sections 5.4–5.7 they are denoted by $\mathbf{U}$ to follow the established tradition coming from probability.

The matrix of second derivatives of a function B on $\mathbb{R}^d$ (the Hessian matrix) is denoted by $\frac{d^2B}{dx^2}$ or H_B. The symbol d^2B stands for the second differential form of B, namely, for the quadratic form $(H_B\,dx, dx)$.