1

Examples of Exact Bellman Functions

1.1 A Toy Problem

Let us start by considering the following simple problem. Suppose we have two positive functions f_1 and f_2 on an interval I, $I \subset \mathbb{R}$, bounded, say, by 1 and having prescribed averages: $\langle f_i \rangle_I = x_i$. We are interested in their scalar product: how large or how small it can be. That is, we would like to find the following two functions:

$$\mathbf{B}^{\max}(x_1, x_2) \stackrel{\text{def}}{=} \sup\{\langle f_1 f_2 \rangle_I : 0 \le f_i \le 1,\ \langle f_i \rangle_I = x_i\} \tag{1.1.1}$$

$$\mathbf{B}^{\min}(x_1, x_2) \stackrel{\text{def}}{=} \inf\{\langle f_1 f_2 \rangle_I : 0 \le f_i \le 1,\ \langle f_i \rangle_I = x_i\} \tag{1.1.2}$$

These functions will be called the Bellman functions of the corresponding extremal problem. In this simple case, the functions can be found by elementary consideration without using any special techniques. Nevertheless, we approach this problem as "a serious one" and provide all the steps in its derivation that we will need in the future consideration of more serious problems.

In what follows, we will consider only the first of these functions, and it will be denoted simply by $\mathbf{B}$ rather than $\mathbf{B}^{\max}$. The first question is about the domain of definition of our function. It is natural to define it on the set of all $x = (x_1, x_2) \in \mathbb{R}^2$ for which there exists at least one pair of test functions f_1 and f_2 such that $\langle f_i \rangle_I = x_i$.

Definition 1.1.1 For a pair of functions $\{f_1, f_2\}$ from $L^1(I)$, we call the point $\mathfrak{b}_{f_1,f_2} \in \mathbb{R}^2$,

$$\mathfrak{b} = \mathfrak{b}_I(f_1, f_2) \stackrel{\text{def}}{=} (\langle f_1 \rangle_I, \langle f_2 \rangle_I),$$

the *Bellman point* of this pair. Very often, the pair of functions is fixed and we are interested in the dependence of the Bellman point on the interval. Then we omit arguments and use only the interval as the index:

$$\mathfrak{b}_J = (\langle f_1 \rangle_{_J}, \langle f_2 \rangle_{_J}) \quad \text{for any interval } J,\ J \subset I\,.$$

Clearly, the Bellman points of all admissible pairs fill the square

$$\Omega = \{x = (x_1, x_2) \colon 0 \le x_i \le 1\}\,.$$

Of course, function $\mathbf{B}$ is formally defined outside the square Ω as well, but it is not interesting to consider this function there because the supremum of the empty set is $-\infty$. Let us state this assertion as a formal proposition. It is trivial in this case, but it might not be so trivial for a more serious problem.

Proposition 1.1.2 (Domain of Definition) *The function* $\mathbf{B}$ *is defined on the domain* Ω.

Proof On the one hand, for any pair of test functions f_1, f_2, we have $0 \le \langle f_i \rangle_{_I} \le 1$, i.e., $\mathfrak{b}_I(f_1, f_2) \in \Omega$. On the other hand, for any $x \in \Omega$, the pair of constant functions $f_i \equiv x_i$ is an admissible pair and $\mathfrak{b}_I(x_1, x_2) = x$. □

Proposition 1.1.3 (Independence on the Interval) *The function* $\mathbf{B}$ *does not depend on the interval* I, *where the test functions are defined.*

Proof Indeed, if we have two intervals I_1 and I_2, then the linear change of variables maps the set of test functions from one interval to another preserving all averages. Therefore, for both intervals, the supremum in the definition of the Bellman function is taken over by the same set. □

We know the values of our function on the boundary $\partial\Omega$.

Proposition 1.1.4 (Boundary Conditions)

$$\begin{aligned} \mathbf{B}(0, x_2) &= 0, \qquad & \mathbf{B}(1, x_2) &= x_2, \\ \mathbf{B}(x_1, 0) &= 0, \qquad & \mathbf{B}(x_1, 1) &= x_1. \end{aligned} \tag{1.1.3}$$

Proof We easily know the boundary values because for these points, the set, over which supremum in the definition of the Bellman function is taken, consists of only one element. Indeed, if $\langle f_i \rangle_{_I} = 0$, then $f_i = 0$ almost everywhere (because $f_i \ge 0$), and therefore, $\langle f_1 f_2 \rangle_{_I} = 0$. If $\langle f_i \rangle_{_I} = 1$, then $f_i = 1$ almost everywhere (because $f_i \le 1$), and hence, $\langle f_i f_j \rangle_{_I} = x_j$. □

Our function possesses an additional symmetry property:

PROPOSITION 1.1.5 (Symmetry)

$$\mathbf{B}(x_1, x_2) = \mathbf{B}(x_2, x_1)\,. \tag{1.1.4}$$

PROOF We can interchange the roles of f_1 and f_2 without changing the value of $\langle f_1 f_2\rangle_{_I}$. Then we interchange x_1 and x_2 keeping the value of the Bellman function stable. □

PROPOSITION 1.1.6 (Main Inequality) *For every pair of points $x^\pm$ from Ω and every pair of positive numbers $\alpha^\pm$ such that $\alpha^- + \alpha^+ = 1$, the following inequality holds:*

$$\mathbf{B}(\alpha^- x^- + \alpha^+ x^+) \ge \alpha^- \mathbf{B}(x^-) + \alpha^+ \mathbf{B}(x^+). \tag{1.1.5}$$

PROOF Let us split the interval I into two parts: $I = I^- \cup I^+$ such that $|I^\pm| = \alpha^\pm |I|$. The integral in the definition of $\mathbf{B}$ can be presented as a sum of two integrals, the first over I^- and the second over I^+:

$$\int_I f_1(s) f_2(s)\,ds = \int_{I^-} f_1(s) f_2(s)\,ds + \int_{I^+} f_1(s) f_2(s)\,ds.$$

After dividing over $|I|$ we get

$$\langle f_1 f_2\rangle_{_I} = \alpha^- \langle f_1 f_2\rangle_{_{I^-}} + \alpha^+ \langle f_1 f_2\rangle_{_{I^+}}.$$

Now, using the independence of the Bellman function on the interval (Proposition 1.1.3), we choose functions $f_i^\pm$ on the intervals $I^\pm$ such that they almost give us the supremum in the definition of $\mathbf{B}(x^\pm)$, i.e.,

$$\langle f_1^\pm f_2^\pm\rangle_{_{I^\pm}} \ge \mathbf{B}(x^\pm) - \eta,$$

for a fixed small $\eta > 0$. Then for the functions $f_i(s)$, $i = 1, 2$, on I, defined as f_i^+ on I^+ and f_i^- on I^-, we obtain the inequality

$$\langle f_1 f_2\rangle_{_I} \ge \alpha^- \mathbf{B}(x^-) + \alpha^+ \mathbf{B}(x^+) - \eta. \tag{1.1.6}$$

Observe that the pair of the compounded functions f_i is an admissible pair of test function corresponding to the point $x = \alpha^- x^- + \alpha^+ x^+$. Indeed, $x^\pm = \mathfrak{b}_{I^\pm}(f_1^\pm, f_2^\pm) = \mathfrak{b}_{I^\pm}(f_1, f_2)$, and therefore,

$$\mathfrak{b}_I(f_1, f_2) = \alpha^- \mathfrak{b}_{I^-}(f_1, f_2) + \alpha^+ \mathfrak{b}_{I^+}(f_1, f_2) = \alpha^- x^- + \alpha^+ x^+ = x.$$

The inequality $0 \le f_i \le 1$ is clearly fulfilled as well. So, we can take supremum in (1.1.6) over all admissible pairs of functions. This yields

$$\mathbf{B}(x) \ge \alpha^- \mathbf{B}(x^-) + \alpha^+ \mathbf{B}(x^+) - \eta,$$

which proves the main inequality because η is arbitrarily small. □

PROPOSITION 1.1.7 (Obstacle Condition)

$$\mathbf{B}(x) \geq x_1 \cdot x_2. \tag{1.1.7}$$

PROOF Since the constant functions $f_i = x_i$ belong to the set of admissible test functions corresponding to the point x, we come to the desired inequality $\sup\{\langle f_1 f_2\rangle : \langle f_i\rangle_{_I} = x_i\} \geq \langle x_1 x_2\rangle_{_I} = x_1 x_2$. □

Before stating the next proposition, we introduce some notation. Let $\mathcal{I}$ be a family of subintervals of an interval I with the following properties:

- $I \in \mathcal{I}$;
- if $J \in \mathcal{I}$, then there is a couple of almost disjoint intervals $J^{\pm}$ (i.e., with the disjoint interiors), such that $J = J^- \cup J^+$;
- $\mathcal{I} = \cup_{n\geq 0}\mathcal{I}_n$, where $\mathcal{I}_0 = \{I\}$, $\mathcal{I}_{n+1} = \{J^-, J^+ \colon J \in \mathcal{I}_n\}$;
- $\lim_{n\to\infty} \max\{|J| \colon J \in \mathcal{I}_n\} = 0$.

If the family $\mathcal{I}$ satisfies the following additional condition

- $|J^-| = |J^+|$,

it is called *dyadic*. For the dyadic family of subintervals, we use notation $\mathcal{D}(I)$ instead of $\mathcal{I}$.

PROPOSITION 1.1.8 (Bellman Induction) *If B is a continuous function on the domain Ω satisfying the main inequality* (*that is just concavity condition*) *and obstacle condition* (1.1.7), *then* $\mathbf{B}(x) \leq B(x)$.

PROOF Fix an interval I and its splitting $\mathcal{I}$. Take an arbitrary point $x \in \Omega$ and two test function f_1 and f_2 on I, $0 \leq f_i \leq 1$, such that $x = \mathfrak{b}_I(f_1, f_2)$. We can rewrite the main inequality in the form

$$|J|B(\mathfrak{b}_J)| \geq |J^+|B(\mathfrak{b}_{J^+}) + |J^-|B(\mathfrak{b}_{J^-}).$$

Let us take the sum of the earlier inequalities when J runs over $\mathcal{I}_k$, the set of subintervals of kth generation. Then $J^{\pm}$ are all intervals of the set $\mathcal{I}_{k+1}$, and we get

$$\sum_{J\in\mathcal{I}_k} |J|B(\mathfrak{b}_J) \geq \sum_{J\in\mathcal{I}_{k+1}} |J|B(\mathfrak{b}_J).$$

Therefore,

$$|I|B(x)| = |I|B(\mathfrak{b}_I) = \sum_{J\in\mathcal{I}_0} |J|B(\mathfrak{b}_J) \geq \sum_{J\in\mathcal{I}_n} |J|B(\mathfrak{b}_J) = \int_I B(x^{(n)}(s))\,ds,$$

where $x^{(n)}$ is a step function defined in the following way: $x^{(n)}(s) = \mathfrak{b}_J$, when $s \in J$, $J \in \mathcal{I}_n$.

We know that $x^{(n)}(s) \to (f_1(s), f_2(s))$ almost everywhere by the Lebesgue differentiation theorem. Since B is continuous, we have $B(x^{(n)}(s)) \to B(f_1(s), f_2(s))$. Now, using the obstacle condition (1.1.7) and the Lebesgue dominated convergence theorem, we can pass to the limit in the obtained inequality as $n \to \infty$.

$$|I|B(x) \geq \int_I B(f_1(s), f_2(s))\, ds \geq \int_I f_1(s) f_2(s)\, ds = |I|\langle f_1 f_2 \rangle_I. \quad (1.1.8)$$

Taking supremum in this inequality over all admissible pairs f_1, f_2 with $\mathfrak{b}_I(f_1, f_2) = x$, we come to the desired estimate. □

According to this proposition, every concave function satisfying the obstacle condition gives us an upper estimate of the functional under consideration. If we are interested in a sharp estimate, we need to look for minimal possible such functions. Due to the symmetry (see Proposition 1.1.5), it is sufficient to consider $x_1 \leq x_2$.

On a triangle, we know our function at the vertices: $\mathbf{B}(0,0) = 0$, $\mathbf{B}(0,1) = 0$, and $\mathbf{B}(1,1) = 1$. The minimal possible concave function passing through the given three points is a linear function. In our case, it is the function $B(x) = x_1$. By the symmetry on the whole square Ω, we get the following *Bellman candidate*[1] $B(x) = \min\{x_1, x_2\}$.

In fact, we have already found the Bellman function.

Theorem 1.1.9

$$\mathbf{B}(x) = \min\{x_1, x_2\}.$$

Proof First of all, by Proposition 1.1.8, the upper estimate $\mathbf{B}(x) \leq B(x)$ is true because B is concave and $\min\{x_1, x_2\} \geq x_1 x_2$. Since there is no concave function satisfying the required boundary condition and that is less than B, we get $\mathbf{B} = B$.

However, in a more difficult problem, it is not so clear that the Bellman candidate cannot be diminished. By this reason, we demonstrate on this example how we will typically prove the lower estimate $\mathbf{B}(x) \geq B(x)$. To this end for every point $x \in \Omega$, we present an admissible test function, realizing the supremum in the definition of the Bellman function. In some papers, such a function (in our case, it is a pair of functions) is called an *extremizer*, but in other papers it is called an *optimizer*. We shall use both these words as synonyms. In our case, the possible pair of extremizers is very

[1] Such a term is used for a function possessing the necessary properties of the Bellman function, e.g., concavity, symmetry, boundary values, etc. After a Bellman candidate is presented, we need to check that it indeed is the desired Bellman function.

simple: $f_i = \mathbf{1}_{[0,x_i]}$. We evidently have $\langle f_i\rangle_{[0,1]} = x_i$ and $\langle f_1 f_2\rangle_{[0,1]} = \min\{x_i\}$. Since by definition $\mathbf{B}(x)$ is the supremum of $\langle f_1 f_2\rangle_{[0,1]}$, when f_i runs over all admissible pairs corresponding to the point x, $\mathbf{B}(x)$ is not less that this particular value, which is equal to $\min\{x_i\} = B(x)$. □

At the end of this section, we would like to explain how to find the extremizers mentioned earlier. Look at the proof of Proposition 1.1.8. Let us take $B = \mathbf{B}$ in this chain of inequalities choosing at the beginning f_1, f_2 to be a pair of extremizers. Since the first and the last terms in the chain of inequalities (1.1.8) are equal, namely, they are $|I|\mathbf{B}(x)$, we must have equalities in each step. In other words, we need to choose such a splitting $x = \alpha^- x^- + \alpha^+ x^+$ to have equality rather than inequality in (1.1.5). In our case, it is easy to do because our Bellman candidate is a concatenation of two linear functions, and if we deal only with one of these linear functions, we always have equality in (1.1.5). Based on this reason, in this simple situation, we can choose extremizers in an almost arbitrary way; the only condition is that all three points x and $x^\pm$ must be in the same triangle: either in $\{x\colon x_1 \le x_2\}$ or in $\{x\colon x_1 \ge x_2\}$.

Let us construct a pair of optimizers for some point x with $x_1 \le x_2$. First we draw the straight line passing through the points x and $x^- \stackrel{\text{def}}{=} (1,1)$. It intersects the boundary of Ω at the point $(0, \frac{x_2 - x_1}{1 - x_1}) \stackrel{\text{def}}{=} x^+$. So, we have $x = x_1 \cdot x^- + (1 - x_1)\cdot x^+$, i.e., $\alpha^- = x_1$, $\alpha^+ = 1 - x_1$, and we need to split our initial interval I (take $I = [0,1]$) in the union $I^- = [0, x_1]$ and $I^+ = [x_1, 1]$. The point $x^- = (1,1)$ is the Bellman point of the only pair $f_1 = f_2 = 1$, hence on $[0, x_1]$ we take both extremal functions equal identically to 1. The point $x^+ = (0, \frac{x_2 - x_1}{1 - x_1})$ is the Bellman point, for example, the pair of constant functions, and we can put $f_1 = 0$ and $f_2 = \frac{x_2 - x_1}{1 - x_1}$ on $[x_1, 1]$. It is easy to check whether this pair of functions gives us an extremizer. However, the second function of this extremizer differs from that presented earlier. What to do to get that extremizer? We only have to split I^+ once more, presenting x^+ as the convex combination of $(0,1)$ and $(0,0)$:

$$x^+ = \frac{x_2 - x_1}{1 - x_1}(0,1) + \frac{1 - x_2}{1 - x_1}(0,0), \qquad I^+ = [x_1, 1] = [x_1, x_2] \cup [x_2, 1].$$

The function f_1 is, as before, the zero function on both subintervals, but we have to take f_2 equal to 1 on $[x_1, x_2]$ and equal to 0 on $[x_2, 1]$. In this way, we come to the pair of functions presented earlier.

We would like to provide support now to the readers for whom the latter paragraph remains unclear: you meet such kind of construction (splitting the interval and representing a Bellman point as a convex combination of two (or more) other Bellman points) many times on the pages of this book. We hope that after several repetitions, the construction becomes absolutely clear.

Exercises

PROBLEM 1.1.1 Find the function $\mathbf{B}$ defined for a similar problem, where the restriction $0 \le f_i \le 1$ is replaced by $|f_i| \le 1$

PROBLEM 1.1.2 Find the function $\mathbf{B}^{\min}$ defined in (1.1.2).

PROBLEM 1.1.3 Find the function $\mathbf{B}^{\min}$ for the set of test functions described in Problem 1.1.1.

1.2 Buckley Inequality

For an interval I and a number $r > 1$, the symbol $A_\infty(I, r)$ denotes the r-"ball" in the Muckenhoupt class A_∞:

$$A_\infty(I,r) \stackrel{\text{def}}{=} \Big\{ w\colon w \in L^1(I),\ w \ge 0,\ \langle w \rangle_J \le r e^{\langle \log w \rangle_J}\ \forall J \subset I \Big\}. \tag{1.2.1}$$

We denote by $\mathcal{D}(I)$ the set of all dyadic subintervals of I and by $A^d_\infty(I,r)$ the dyadic analog of (1.2.1), i.e., in the definition of $A^d_\infty(I,r)$, we consider only $J \in \mathcal{D}(I)$.

THEOREM (Buckley [**19**]) *There exists a constant $c = c(r)$ such that*

$$\sum_{J \in \mathcal{D}(I)} |J| \Big(\frac{\langle w \rangle_{J+} - \langle w \rangle_{J-}}{\langle w \rangle_J} \Big)^2 \le c(r)|I|$$

for any weight w from $A^d_\infty(I,r)$.

Now, we are ready to introduce the main object of our consideration, the so-called Bellman function of the problem.

$$\begin{aligned} \mathbf{B}(x) &= \mathbf{B}(x_1, x_2; r) \\ &\stackrel{\text{def}}{=} \sup_{w \in A^d_\infty(I,r)} \left\{ \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} |J| \left(\frac{\langle w \rangle_{J+} - \langle w \rangle_{J-}}{\langle w \rangle_J} \right)^2 : \right. \\ &\qquad\qquad \left. \langle w \rangle_I = x_1,\ \langle \log w \rangle_I = x_2 \right\}. \end{aligned} \tag{1.2.2}$$

Let us note that we did not assign the index I to $\mathbf{B}$ despite the fact that all test functions w in its definition are considered on I. This omission is not due to our desire to simplify notation, but rather an indication of the very important fact that the function $\mathbf{B}$ does not depend on I; Proposition 1.1.3 holds in this situation by the same reason.

For a given weight $w \in A_\infty^d(I,r)$, we introduce a Bellman point $\mathfrak{b}_I(w)$ in the following way: $\mathfrak{b}_I(w) = (\langle w\rangle_{\scriptscriptstyle I}, \langle \log w\rangle_{\scriptscriptstyle I})$. Note that for all admissible weights and for any dyadic subinterval $J \subset I$, the corresponding Bellman point $\mathfrak{b}_J(w)$ is in the following domain Ω_r:

$$\Omega_r \stackrel{\text{def}}{=} \left\{x = (x_1, x_2)\colon \; \log\frac{x_1}{r} \le x_2 \le \log x_1\right\}.$$

Indeed, the right bound is simply Jensen's inequality and the left one is fulfilled because our weight w is from $A_\infty^d(I,r)$.

To show that Ω_r is the domain of the function $\mathbf{B}$, we need to check that for any point $x \in \Omega_r$ there exists an admissible weight with $\mathfrak{b}_I(w) = x$. However, we leave this for the reader as an exercise (see Problem 1.2.1).

Now we prove the crucial property of the function $\mathbf{B}$ that follows directly from its definition.

Lemma 1.2.1 (Main Inequality) *For every pair of points $x^\pm$ from Ω_r such that their mean $x = (x^+ + x^-)/2$ is also in Ω_r, the following inequality holds:*

$$\mathbf{B}(x) \ge \frac{\mathbf{B}(x^+) + \mathbf{B}(x^-)}{2} + \left(\frac{x_1^+ - x_1^-}{x_1}\right)^2. \tag{1.2.3}$$

Proof Let us split the sum in the definition of $\mathbf{B}$ into three parts: the sum over $\mathcal{D}(I^+)$, the sum over $\mathcal{D}(I^-)$, and an additional term corresponding to I itself:

$$\begin{aligned}\frac{1}{|I|}&\sum_{J\in\mathcal{D}(I)} |J| \left(\frac{\langle w\rangle_{\scriptscriptstyle J+} - \langle w\rangle_{\scriptscriptstyle J-}}{\langle w\rangle_{\scriptscriptstyle J}}\right)^2 \\ &= \frac{1}{2|I^+|}\sum_{J\in\mathcal{D}(I^+)} |J| \left(\frac{\langle w\rangle_{\scriptscriptstyle J+} - \langle w\rangle_{\scriptscriptstyle J-}}{\langle w\rangle_{\scriptscriptstyle J}}\right)^2 \\ &\quad + \frac{1}{2|I^-|}\sum_{J\in\mathcal{D}(I^-)} |J| \left(\frac{\langle w\rangle_{\scriptscriptstyle J+} - \langle w\rangle_{\scriptscriptstyle J-}}{\langle w\rangle_{\scriptscriptstyle I}}\right)^2 \\ &\quad + \left(\frac{\langle w\rangle_{\scriptscriptstyle I+} - \langle w\rangle_{\scriptscriptstyle I-}}{\langle w\rangle_{\scriptscriptstyle I}}\right)^2.\end{aligned}$$

Using the fact that $\mathbf{B}$ does not depend on the interval where the test functions are defined, we can choose two weights $w^{\pm}$ on the intervals $I^{\pm}$ that almost give us the supremum in the definition of $\mathbf{B}(x^{\pm})$, i.e.,

$$\frac{1}{|I^{\pm}|}\sum_{J\in\mathcal{D}(I^{\pm})}|J|\left(\frac{\langle w^{\pm}\rangle_{J+}-\langle w^{\pm}\rangle_{J-}}{\langle w^{\pm}\rangle_{J}}\right)^2\geq \mathbf{B}(x^{\pm})-\eta,$$

for an arbitrary fixed small $\eta>0$. Then for the weight w on I, defined as w^+ on I^+ and w^- on I^-, we obtain the inequality

$$\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)}|J|\left(\frac{\langle w\rangle_{J+}-\langle w\rangle_{J-}}{\langle w\rangle_{J}}\right)^2\geq \frac{\mathbf{B}(x^+)+\mathbf{B}(x^-)}{2}-\eta+\left(\frac{x_1^+-x_1^-}{x_1}\right)^2. \tag{1.2.4}$$

Observe that the compound weight w is an admissible weight, corresponding to the point x. Indeed, $x^{\pm}=\mathfrak{b}_{I^{\pm}}(w)$ and by the construction of $w^{\pm}$ we have $w^{\pm}\in A_\infty^d(I^{\pm},r)$. Therefore, the weight w satisfies the inequality $\langle w\rangle_J\leq re^{\langle\log w\rangle_J}$ for all $J\in\mathcal{D}(I^+)$, since w^+ does, and for all $J\in\mathcal{D}(I^-)$, since w^- does. Lastly, $\langle w\rangle_I\leq re^{\langle \log w\rangle_I}$, because, by assumption, $x\in\Omega_r$.

We can now take supremum in (1.2.4) over all admissible weights w, which yields

$$\mathbf{B}(x)\geq\frac{\mathbf{B}(x^+)+\mathbf{B}(x^-)}{2}-\eta+\left(\frac{x_1^+-x_1^-}{x_1}\right)^2.$$

This proves the main inequality because η is arbitrarily small. □

Lemma 1.2.2 (Boundary Condition)

$$\mathbf{B}(x_1,\log x_1)=0.$$

Proof Let us take a boundary point x of our domain Ω_r, that is a point with $x_2=\log x_1$. Since the equality in Jensen's inequality $e^{\langle w\rangle}\leq\langle e^w\rangle$ occurs only for constant functions w, the only test function corresponding to x is the constant (up to a set of measure zero) weight $w=x_1$. So, on this boundary, we have $\mathbf{B}(x)=0$. □

Lemma 1.2.3 (Homogeneity) *There is a function g on $[1,r]$ satisfying $g(1)=0$ and such that*

$$\mathbf{B}(x)=\mathbf{B}(x_1e^{-x_2},0)=g(x_1e^{-x_2}).$$

Proof For a weight w on an interval I and a positive number τ, consider a new weight $\tilde{w}=\tau w$. If $x=\mathfrak{b}_I(w)$, i.e., $x_1=\langle w\rangle_I$, $x_2=\langle\log w\rangle_I$, then for the point $\mathfrak{b}_I(\tilde{w})=\tilde{x}$ we have $\tilde{x}_1=\tau x_1$, $\tilde{x}_2=x_2+\log\tau$. Note that the expression in the definition of $\mathbf{B}$ is homogeneous of order 0 with respect to w, i.e., it does not depend on τ. Since the weights w and $\tilde{w}$ run over the whole set $A^d_\infty(I,r)$ simultaneously, we get $\mathbf{B}(x) = \mathbf{B}(\tilde{x})$. Choosing $\tau = e^{-x_2}$, we obtain

$$\mathbf{B}(x) = \mathbf{B}(x_1 e^{-x_2}, 0).$$

To complete the proof, it suffices to take $g(s) = \mathbf{B}(s,0)$. The boundary condition $g(1) = 0$ holds due to Lemma 1.2.2. □

We are now ready to demonstrate how the Bellman induction works in this case.

Lemma 1.2.4 (Bellman Induction) *Let B be a nonnegative function on Ω_r satisfying the main inequality in Ω_r (Lemma 1.2.1). Then*

$$\mathbf{B}(x) \le B(x).$$

Proof Fix an interval I and a point $x \in \Omega_r$. Take an arbitrary weight $w \in A^d_\infty(I,r)$ such that $\mathfrak{b}_I(w) = x$. Let us repeatedly use the main inequality in the form

$$|J|\,B(\mathfrak{b}_J) \ge |J^+|\,B(\mathfrak{b}_{J+}) + |J^-|\,B(\mathfrak{b}_{J-}) + |J| \left(\frac{\langle w\rangle_{J+} - \langle w\rangle_{J-}}{\langle w\rangle_J} \right)^2,$$

applying it first to I, then to the intervals of the first generation (that is $I^\pm$), and so on until $\mathcal{D}_n(I)$:

$$\begin{aligned} |I|\,B(\mathfrak{b}_I) &\ge |I^+|\,B(\mathfrak{b}_{I+}) + |I^-|\,B(\mathfrak{b}_{I-}) + |I| \left(\frac{\langle w\rangle_{I+} - \langle w\rangle_{I-}}{\langle w\rangle_I} \right)^2 \\ &\ge \sum_{J\in\mathcal{D}_n(I)} |J|\,B(\mathfrak{b}_J) + \sum_{k=0}^{n-1} \sum_{J\in\mathcal{D}_k(I)} |J| \left(\frac{\langle w\rangle_{J+} - \langle w\rangle_{J-}}{\langle w\rangle_J} \right)^2. \end{aligned}$$

Therefore,

$$\sum_{k=0}^{n-1} \sum_{J\in\mathcal{D}_k(I)} |J| \left(\frac{\langle w\rangle_{J+} - \langle w\rangle_{J-}}{\langle w\rangle_J} \right)^2 \le |I|\,B(\mathfrak{b}_I),$$

and passing to the limit as $n \to \infty$, we get

$$\sum_{J\in\mathcal{D}(I)} |J| \left(\frac{\langle w\rangle_{J+} - \langle w\rangle_{J-}}{\langle w\rangle_J} \right)^2 \le |I|\,B(x).$$

Taking supremum over all admissible weight w corresponding to the point x, we come to the desired estimate. □