The Black–Scholes–Merton Model as an Idealization of Discrete-Time Economies

This book examines whether continuous-time models in frictionless financial economies can be well approximated by discrete-time models. It specifically looks to answer the question: In what sense and to what extent does the famous Black–Scholes–Merton (BSM) continuous-time model of financial markets idealize more realistic discrete-time models of those markets? While it is well known that the BSM model is an idealization of discrete-time economies where the stock price process is driven by a binomial random walk, it is less known that the BSM model idealizes discrete-time economies whose stock price process is driven by more general random walks. After recounting the foundations of discrete- and continuous-time models of financial markets, David M. Kreps develops the general theory of convergence of discrete-time models to their continuous-time limits, both in mathematical and economic terms. The exposition lowers the entry barriers to the literature of financial mathematics for less-technical readers, while providing a fuller understanding of the connections between BSM and nearby discrete economies.

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Preface

I began this monograph (which, at the time, was a nascent paper) with the objective of understanding how and how well continuous-time models of economic phenomena – and in particular models that employ Brownian motion – compare with “near by” discrete-time models. We know that the comparisons can sometimes be strained; see, for instance, Fudenberg and Levine (2009) and Sadzik and Stacchetti (2015). So, it seemed to me, a general theory connecting the two types of models ought to be available.

I still believe that such a general theory is available. But I quickly discovered that it is difficult, and so to build my intuition, I did what any faculty member in a school of management would do: I began to look at “case studies.” In particular, I began with the connections between discrete- and continuous-time models of financial markets and, in particular, the connections between the famous Black–Scholes–Merton\(^1\) theory of pricing contingent claims by arbitrage and discrete-time models – the seminal reference is Cox, Ross, and Rubinstein (1979) – that, in the limit, “converge” to the Black–Scholes–Merton (BSM, for short) model. The scare quotes around “converge” signify that the point of the exercise is to understand what convergence means in this context and, in particular, to understand how general is this convergence.

In the course of studying these issues, I first discovered that, in some regards, BSM is the economic “limit” of more discrete-time models than I had originally thought – developing a precise statement of what this means is one of the main points of this monograph – and then that, while this fact was not known by me and, seemingly, is not well known by many of colleagues in the field of Finance, it is well known to scholars – primarily mathematicians who specialize in probability – who are members of the community of so-called Financial Mathematicians. I believe that what is known to this community of scholars should be well-known more broadly to financial economists, and so I embarked on writing an exposition of those ideas in the simplest possible settings and in a single, unified presentation, so they would be accessible to mainstream financial

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\(^1\) It is common in the literature to refer to the Black–Scholes theory, the Black–Scholes Option Pricing Model, and the Black–Scholes Formula. But the history of these ideas supports giving equal credit to Robert Merton. Hence, in this monograph, I refer to the Black–Scholes–Merton model and theory.
economists. This monograph is the result.

To be clear, “in the simplest possible setting” does not mean “in a simple setting.” The limit is the BSM model of an economy, which is mathematically a complex thing. And, to give precise limiting results, I must employ some mathematically sophisticated concepts and results, among which are weak convergence of probabilities on spaces of functions, the classic functional central limit theorem of Donsker, and the Skorohod Representation Theorem. Many mainstream financial economists (and economists more generally) will not find this an easy read, although as each of these tools is introduced, I provide analogies to results on the real line that most readers will know.

At the same time, this is not an archive of everything known on these issues. To make this more accessible, I do not give general results, nor do I pose the results in the mathematically most pleasing settings. (In particular, experts will be dismayed that I conduct business in $C[0, 1]$ rather than in $D[0, 1]$.) My hope is that the exposition will bring less-technical readers along to (at least) a good understanding of the connections between BSM and near by discrete-time economies.

Website

While I hope and expect that this monograph contains only a few errata, it is inevitable that there are some. I may have missed pertinent references, about which readers will want to know. There are several questions and conjectures left open, which (I hope) I or others will resolve. And, of course, work on the topics discussed here is ongoing. For these reasons, a public website has been created at the URL discrete2continuous.stanford.edu at which you can find supplementary materials of these sorts. I recommend visiting this website, to see what is posted there, as you read this book. (If you discover errors or typos, or if you wish to suggest additional references, and in particular if you can resolve some of the questions I’ve left open, please email me at kreps@stanford.edu.)

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gestions that greatly improved the final version. Walter Schachermayer was both generous and invaluable on technical aspects, most significantly concerning Proposition 5.1b, but concerning many other matters, as well. Assistance as well from David Aldous, Rama Cont, Darrell Duffie, Hans Föllmer, Michael Harrison, Friedrich Hubalek, Jean-Luc Prigent, and Ruth Williams is gratefully acknowledged. Two excellent anonymous reviewers provided further assistance.

Jeff Ely was initially the Econometric Society Monograph Series editor handling this book; midstream, Andrea Pratt took over. I am grateful to both of them for the expeditious handling of the process. Karen Maloney was the acquisition editor for Cambridge University Press; she, and her editorial assistant Rachel Blaifeder, had to deal with an author who has a well deserved reputation for intransigence when it comes to book production; they did so with grace and patience. Although this book is not within his normal purview in terms of subject matter, David Tranah of CUP was very helpful concerning some the technical details related to production. Adam Hooper did an excellent job coordinating production, and copyeditor Anne Valentine exhibited a combination of flexibility to the author’s desires where it made sense and strength in insisting on flexibility from the author when that was required. The book is significantly improved in style and readability through her efforts.

Dedication

The economic ideas in this monograph are due to a number of authors, and acknowledgements are given as appropriate. But the basic insight about how these questions can be approached is due to Stephen A. Ross, whose untimely death removed from financial economics one of the giants of the field. I dedicate this monograph to his memory.