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About this book

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This book consists of 23 invited, original peer-reviewed papers in Combinatorial Game Theory (CGT) [5; 11; 46][1] — seven surveys and sixteen research papers. This is the fifth volume in the subseries Games Of No Chance (GONC) of the Mathematical Sciences Research Institute Publications. The name emphasizes these volumes' focus on play with no dice and no hidden cards, situating them in the landscape of game theory at large, where incomplete and/or imperfect information is common. Considering our class of games, *perfect play* can in theory be computed, and thus we include games such as CHESS, GO[2] and CHECKERS, but not YAHTZEE, BACKGAMMON and POKER.

Another characterizing feature is that combinatorial games are usually zero-sum, typically win-loss situations, although in some games draws are also possible. Players alternate in moving, so for any game description, we include a move flag of who starts. Game positions can be very sensitive to this move flag, and a common question is, given a combinatorial game, if I offer you to start, should you accept?

Often it is better to start, but not always. In the popular game of GO, the second player is rewarded a "komi" advantage of about 6.5 points before the game begins. In CHESS it is also regarded that White has a slight advantage. In neither of these games there is a mathematical proof, of this believed advantage, but since the games have been played for thousands of years, the belief seems well founded through overwhelming play-evidence.

[1]This book was initiated at the Combinatorial Game Theory Workshop, January 2011, at the Banff International Research Station (BIRS). As usual, this workshop attracted many researchers from Asia, Europe and North America, and it was organized by Richard Nowakowski (Dalhousie University), Elwyn Berlekamp (University of California, Berkeley), Aviezri Fraenkel (Weizmann Institute of Science), Martin Mueller (University of Alberta), and Tristan Cazenave (Paris-Dauphine University).

[2]On page 8, Carlos Santos reviews briefly DeepMind's AI advances of AlphaZero, a generalization of AlphaGo Zero, which recently beat the previously highest ranked CHESS program Stockfish, after just a few hours of training, alas using a massive computing power.

There are play-games which are also math-games.[3] The first player loses the game TWENTYONE. The rules are as follows: start with the number 21. The players alternate in subtracting 1 or 2 from the current number. If you start, then (in perfect play) the other player will "complement your move modulo 3", and win after 7 such rounds. Here, the game is specially *designed* to be a second player win.

We include three papers (13 on p. 313, 14 on p. 333 and 19 on p. 403) in the spirit of mechanism design in game theory; here, given a candidate set of P-positions[4], related to Beatty's classical theorem [2; 3], these contributions construct three classes of game rules with this set as the set of P-positions. The problem originates in the traditional combinatorial game of WYTHOFF NIM [53], which has a Beatty type solution on the modulus the Golden section. Generalizations of this game have begun to accumulate a lot of work, and we present the first comprehensive survey related to the heritage of Wilhelm Abraham Wythoff (paper 2, p. 35).

Singmaster famously proved [48] that almost no combinatorial game is a second player win. In this volume, we have a contribution by Singmaster (paper 7, p. 207), where he surveys the history of binary arithmetic in connection with puzzles, that is "one-player games" such as the CHINESE RINGS and the TOWER OF HANOI [27].

The game of HEX [25] is a classical game related to Brouwer's fixed-point theorem: two players compete in being the first to connect opposite sides of a hexagonal grid. A convention is often added, to compensate for the first player's advantage, namely, immediately after the first move the second player is given the opportunity to switch players. We have an amazing contribution of the theory of HEX in this book, by its current master, Ryan Hayward (paper 17, p. 387).

The game of CHOMP [9] has become famous for various reasons. Two players alternate to "chomp" pieces from a chocolate bar, by pointing at one piece and

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[3]This is an informal distinction, but can be helpful to some extent: typical play-games are GO, CHESS, HEX, CHECKERS, KONANE, BLOKUS, FOUR IN A ROW, MANCALA, TIC TAC TOE, DOTS&BOXES, FOX&GEESE etc, while typical Math-games are DOMINEERING, HACKENBUSH, NIM, HEX, FOX&GEESE, WYTHOFF NIM, EUCLID, FIBONACCI NIM, etc. So, for example HEX and FOX&GEESE belong to both classes. One way of classification is to use the literature to claim membership of the latter class, while membership in the former class is due to being a popular social game. Other, more formal ways of classification may be suitable depending on purpose, but at least play-games should exclude games of values such as OMEGA; on the other hand many loopy/cyclic games are perfectly playable for human players. The distinction can be important when we aspire to build game rules, knowing beforehand "the solution"; see discussion on page 12 related to papers 13, 14 and 19 in this book.

[4]The class of second player win positions is usually called P-positions (*P*revious player wins), and they can be recursively computed for a game with a finite number of positions, starting with the terminal(s).

eating everything to the right and above, trying to avoid eating the lower left poisonous piece. The first player has a winning move, and the argument is as follows. The first player chomps the single right uppermost piece.[5] Now, any move the second player makes, could instead have been played also by the first player. Therefore the second player cannot have a winning strategy in this game. Of course, this argument does not give a clue of how to play. No general rigorous method is known, even for three row CHOMP, but to great surprise a method in physics, called renormalization, gives an estimate of where the first winning move must be [20]. This result, among others, were also published in a previous volume in this series (3.349 in the Index). Here, we are happy to report yet another finding where the renormalization approach gives precise estimates of solutions of a novel generalization of WYTHOFF NIM, called LINEAR NIMHOFF (paper 15, p. 343), indicating that in an infinite class of "linear" Wythoff extensions, all second player winning positions are distributed along thin lines.

The theory of combinatorial games was initiated by Charles Leonard Bouton in 1901 [8], when he discovered that elementary binary arithmetics solves the game of NIM, in a way that any finite number of heaps can be replaced by one single heap. For the rules of this game, see page 171 in Siegel's article (paper 6, p. 169). The next major development was in the 1930s, when Roland Percival Sprague [50] and Patrick Michael Grundy [23] independently discovered that any impartial 2-player game with the normal ending convention (last move wins) is equivalent to a one heap NIM position (and this development is also surveyed in Siegel's chapter). Again, observe how brilliant and surprising this is. Played on its own, of course, the one heap NIM game is a ridiculous thing — if the heap is nonempty, you win by removing all pebbles, and otherwise you already lost — but, by playing in a "disjunctive sum" with other NIM heaps, then this simple game encodes *any* other game in its class, and the class is huge (!), and moreover, the simple arithmetics of solving NIM then suffices to solve any such game. So, "there is something going on here" (attempting to read the minds of Sprague and Grundy). Although, this class of games is solved in theory, computationally, the games are often hard, and we include a famous yet unsolved problem in this book, presented by Grossman (paper 16, p. 373).

The next big discovery occurred in the 1950s when John Milnor [42] and Olof Hanner [24] developed a similar theory for a wide class of scoring-play games (highest score wins) without zugzwangs, that is games where it is never (!) bad to move first. Although "scoring games" are standard in game theory

[5] It has been conjectured that, in a natural generalization of CHOMP (SUBSET TAKEAWAY), "taking the largest element" is a winning move [22], but more recently counterexamples were found [10]. Note also that, given a quadratic chocolate bar of size larger than 1, you will win if you chomp off all except the lower row and left column.

at large, via concepts such as “utility”, “revenue” etc., in CGT, the main idea usually concerns the “move ability”; when is it beneficial to have move options? In fact Milnor’s games belong to a class of scoring games where either no player can move, or both players can move, so they remain closer to “economic games” for these two reasons.

In this volume, we proudly present Stewart’s eye-opener on the full class of scoring games (paper 22, p. 447), where he pinpoints the difficulty of analyzing the full class; the problem boils down to a subset of games, where you want to start, but you do not have any move option (!). Those readers who study the misère play convention (last move loses) would acknowledge with a nod, that this situation usually induces severe complications. Many standard CGT (normal-play) tools fail.

These type of problems are discussed in three survey papers in this volume: papers 6 (p. 169) and 4 (p. 113) on impartial and partisan misere play developments respectively, and paper 3 (p. 89) on recent progress in scoring-play. In fact, a theory has recently been developed for those scoring games which exclude exactly Stewart’s problematic games, the class of guaranteed scoring games [35; 34], and it is shown that the normal-play games are order embedded into this class. In this landscape, intersecting scoring- with normal-play, we find also the master pieces on DOTS&BOXES [6] and Mathematical GO [7].

The huge leap forward was in the 1970s-80s when Elwyn Berlekamp, John Horton Conway and Richard Kenneth Guy developed the normal-play theory to encompass so called partizan games, where players do not necessarily have the same move options [11; 5]. They adapted Milnor’s definitions of disjunctive sum and game comparison [42], which was inspired by the apparent decomposition of GO positions into independent components towards the end of play.

Let G, H be normal-play games (without draws). Then $G \geq H$ if, for any normal-play game X, Left wins the game $H+X$ then Left wins the game $G+X$.

The intuition is as follows: let us imagine that you (playing Left) are in the middle of a complex game, a game which is decomposed in several (a finite number) of components — you are allowed to play only in one component at the time — and get an offer by a passerby to exchange one of the game components for another one. Let us say, your game is $G+X$, where X denotes a complex part of the game that you do not quite understand, and the passerby offers the game H in exchange for G, both much simpler games. Should you accept this offer?

One of the main theorems of normal-play CGT without draws [11] is that you can ignore the complicated X component, and simply play out the game $G-H=G+(-H)$, where the negative denotes that the players have swapped positions; then check whether you win this game when the opponent starts, which is the same as checking whether $G - H \geq 0$.[6]

In this book we have a unique contribution (paper 10, p. 271), by Carvalho and Santos, where the authors describe a ruleset, a modification of the traditional Hawaiian ruleset KONANE to "PORTUGUESE KONANE", which has a position in each equivalence class of *short* (acyclic games with finite ranks and out-degrees) normal-play games. One could think of this as a CGT analogy of a universal Turing machine: one ruleset encodes it all. This relates to computational complexity. Again, we can imagine CGT relatives to simple universal machines, such as Emil Post's classical Tag productions; many extremely simple rulesets, such as OFFICERS (paper 16, p. 373) has so far defied all solution attempts by human and computer. In fact, recent development in the field contributes three classes of Turing complete classes of combinatorial games [16; 38; 39]. In this volume (paper 9, p. 299), Burke and George show that a generalization of NIM on a graph is PSPACE-complete, a more common hardness measure for combinatorial games [26] (see also 3.3 in the Index).

Some games are cyclic (or loopy), i.e., they have infinite game trees, and such games can be hard to analyze, although FOX&GEESE is an example of a ruleset, where analysis has been fruitful [5].[7] Moreover, one full class of games is fully understood; for the class of loopy normal-play impartial games (on finitely many positions), a complete theory is known. The first solution was given by Smith [49], and then using a more constructive algorithmic approach, Fraenkel and Yesha [19] generalize the classical Sprague–Grundy theory by letting "infinities", enumerate the loopy game values.[8] In this volume (paper 21,

[6]Let us illustrate with an example: the game component is $G = *$, and the game offer is "up" is defined by $H = \uparrow = \{0 \mid *\}$, where the game (class) "$*$" denotes a NIM heap of size one, and where "0" denotes the equivalence class containing as simplest element the empty game, that is the game where no player can move. Suppose that you are playing Left. In this case, you should not exchange G for H. The reason for this is the following: play the game $G - H$, and ask the other player, Right, to start. The negative of "up" is "down", which is the game $-H = \downarrow = \{* \mid 0\}$. That is, the test is to let Right start the game $* + \{* \mid 0\} = \{\{* \mid 0\}, 0 \mid \{* \mid 0\}, *\}$. Right has a good move. Which one?

[7]The original 1982 version of Winning Ways included a lot of examples of loopy games, including subclasses such as stoppers and enders, and more complicated examples, such as BACH'S CAROUSEL. The usual rules of canonical forms still apply to stoppers. In the second edition of Winning Ways, the much-enlarged chapter on FOX&GEESE pretty much solved all initial positions of that game, and many others, thanks also to Siegel's popular program CGSuite [47], much of which he developed in the course of those studies. So loopy games have long played a prominent role in the core content of CGT.

[8]As a personal note by the editor, optimal play may be infinite, but a slightest mistake by your opponent may lead to your easy victory; throughout childhood I played hundreds of games of the traditional game of PICARIA (which was proved drawn in [37]) an extension of THREE MEN MORRIS, both cyclic generalizations of TIC-TAC-TOE, and those plays always concluded with a winner.

p. 439) Sarkar establishes an infinite class of drawn positions of the classical CGT ruleset PHUTBALL, so here, players are indifferent to an offer of choosing side, and playing first or second. See also 3.91 in the Index (and 3.125) for a brilliant introduction to this topic.

We have more pioneers in this book. Some rulesets encourage questions of the form "how many game positions are there?"; this holds true for the class of *placement games* (papers 9 on p. 259, 11 on p. 285 and 12 on p. 297), where relations with simplicial complexes and generating functions are skillfully exposed by Brown et al., Faridi, Huntemann and Nowakowski. This type of games are particularly appealing to "conjoin", which is demonstrated by Huggan and Nowakowski in paper 18 (p. 395). In paper 23 (p. 469), Weimerskirch presents a CGT framework which generalizes the normal and misère conventions and ingeniously includes the notion of disjunctive sum, which brings us back to the heat of the matter; Berlekamp gives a splendid performance in surveying the temperature of the field (paper 1, p. 21). How urgent is it to move in the game component X? He also shows how urgency, and temperature, can be precisely captured by playing the original game in conjunction with an idealized stack of coupons.

The measure of "importance to move" is also captured in the setting of bidding games (paper 20, p. 421), where each play consists in two phases, first make your bid, and if you win the bid you get to move, hence mixing in "auction play" a popular subject in algorithmic game theory to the setting of combinatorial games.

Since its start in the early 1990s, this series of books has captured much of the core of the CGT-development. To celebrate its 20th anniversary, and as suggested by Elwyn Berlekamp, we include an index of all published GONC papers, compiled by Silvio Levy.[9]

An elementary introduction to combinatorial games is contained in the first part of paper 6 (p. 169), by Aaron Siegel, before he plunges into the complexity of the misère quotients and more; see also his current state of the art reference [46], a monumental contribution to the field of combinatorial games.

The book is divided into two sections: Survey articles and Research articles. In the Survey section, we find:

(1) Temperatures of games and coupons (Berlekamp)

(2) Wythoff visions (Duchêne, Fraenkel, Gurvich, Ho, Kimberling, Larsson)

[9]The previous books are *Games of no chance*, volume **29** in the MSRI Publications series (1998), *More games of no chance*, **42** (2002), *Games of no chance 3*, **56** (2009), and *Games of no chance 4*, **66** (2015); all were edited by Richard J. Nowakowski, GONC 3 jointly with Michael H. Albert.

(3) Scoring games: the state of play (Larsson, Nowakowski, Santos)
(4) Restricted developments in partizan misère game theory (Milley, Renault)
(5) Unsolved problems in combinatorial games (Nowakowski)
(6) Misère games and misère quotients (Siegel)
(7) An historical tour of binary and tours (Singmaster)

The Research papers are:

(8) A note on polynomial profiles of placement games (Brown et al.)
(9) A PSPACE-complete Graph Nim (Burke, George)
(10) A nontrivial surjective map onto the short Conway group (Carvalho, Santos)
(11) Games and complexes I: Transformation via ideals (Faridi, Huntemann, Nowakowski)
(12) Games and complexes II: Weight games and Kruskal–Katona type bounds (Faridi, Huntemann, Nowakowski)
(13) Chromatic Nim finds a game for your solution (Fischer, Larsson)
(14) Take-away games on Beatty's theorem and the notion of k-invariance (Fraenkel, Larsson)
(15) Geometric analysis of a generalized Wythoff game (Friedman, Garrabrant, Landsberg, Larsson, Phipps-Morgan). Related to work in these volumes: Friedman, Landsberg (3.349)
(16) Searching for periodicity in Officers (Grossman)
(17) Good pass moves in no-draw HyperHex: two proverbs (Hayward). Related work in these volumes: Anshelevich (2.151); Hayward (3.151); Payne, Robeva (4.207); Henderson, Hayward (4.129)
(18) Conjoined games: Go-cut and Sno-Go (Huggan, Nowakowski)
(19) Impartial games whose rulesets produce given continued fractions, (Larsson, Weimerskirch)
(20) Endgames in Bidding Chess (Larsson, Wästlund). Related work (a.k.a. Richman games) in these volumes: Lazarus, Loeb, Propp, Ullman (1.427, 1.439) (the field initiator); Payne, Robeva (4.207)
(21) Scoring play combinatorial games (Stewart)
(22) Phutball draws (Sarkar). Related work in these volumes: Demaine, Demaine, Eppstein (2.351); Grossman Nowakowski (2.361); Siegel (3.91)
(23) Generalized misère play (Weimerskirch)

Before we move on, we would like to say a few words about the recent blooming development of AI and deep neural networks in playing combinatorial games. Thanks to Carlos Santos for contributing this discussion:

> DeepMind team published an arXiv-preprint (December 5th, 2017) about AlphaZero, a computer program developed to play GO, and generalized to play CHESS (and SHOGI). Within 24 hours, it achieved an outstanding level of play.
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> AlphaZero was trained with no opening theory or endgame tables. Comparing with the previous Monte Carlo algorithms, AlphaZero used just 80 000 positions per second, whereas Stockfish used 70 million. Even so, it won against Stockfish: in 100 games AlphaZero scored 25 wins and 25 draws with White, while with Black it scored 3 wins and 47 draws. It didn't lose a game, with the final score 64:36. The 9th game of the match showed an amazing attacking player with profound positional play. The 10th game was a masterpiece with identical characteristics.
>
> Therefore, as humans know, sometimes less is more! It seems a historical moment, AlphaZero Chess presents a very good "human" CHESS style. But with the incredible power of precise calculations.
>
> Garry Kasparov said "It is a remarkable achievement, even if we should have expected it after AlphaGo."

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