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General Editors

B. BOLLOBÁS, W. FULTON, F. KIRWAN,
P. SARNAK, B. SIMON, B. TOTARO

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Operator Analysis
Hilbert Space Methods in Complex Analysis

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To
Sarah, Suzanne, and *Зинаида*

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Preface

The philosophy of this book is that Hilbert space geometry binds function theory and operator theory together, not only allowing each to aid the other but creating a rich structure that can be used to discover new phenomena. There is a “three-way street” between operator theory and function theory: sometimes one uses function theory to prove operator theorems, sometimes one uses operator theory to prove function theorems, and sometimes the theories are so interwoven that one cannot even state the theorem without using the language of both operator theory and function theory.

The main thrust of the book is to discover and prove theorems about holomorphic functions and complex geometry with the aid of Hilbert space geometry and operator theory. The holomorphic functional calculus permits us to substitute commuting operators for the variables of a holomorphic function, a step that reveals hidden properties of the function—something every function theorist should want to do.

It is remarkable how little operator theory one needs in order to prove significant facts in function theory. There will be no call here for the detailed and subtle theories of particular classes of operators, but we shall make heavy use of the functional calculus for operators. The theories of operator dilations and spectral sets also play an important role. We explain carefully what is required from these theories. The reader may either take this material on trust or consult the references that we give.

Part I of the book is devoted to holomorphic functions on domains in complex Euclidean space, beginning with scalar-valued functions on the unit disc in the complex plane, where intuition is most easily developed. Here the central notion of a Hilbert space *model* of a function is introduced, as are several types of arguments that will recur throughout the book. Gradually we build up to various domains in higher dimensions and to operator-valued functions, the latter being important for engineering applications. We do not

aim for maximum generality, which can come at the cost of a sacrifice of elegance and impact.

Part II concerns *non-commutative functions*, that is, functions of non-commuting variables. This is a topic of relatively recent study, and one that is currently undergoing rapid development. It transpires that the Hilbert space methods of Part I are well suited to this new context, and we derive many analogs of classical theorems.

Our intended audience is graduate students and mathematicians interested in complex function theory and/or operator theory. We do not require familiarity with several complex variables, but we do assume that the reader has a basic knowledge of complex analysis and functional analysis.

The moment you buy into the functional calculus, you're ready to roll!

Acknowledgments

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Parts of the book were read in draft form by several people, who caught some mistakes. We are specially grateful to Alberto Dayan, Chris Felder, Michael Hartz, Mark Mancuso, James Pascoe, and Jeet Sampat. All remaining mistakes and typographical errors are, of course, the responsibility of the authors.

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