CAMBRIDGE

Cambridge University Press
978-1-108-48436-7 — Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization
Houman Owhadi , Clint Scovel
Table of Contents
More Information

# Contents

*Preface* *page* xiii
*Acknowledgments* xiv
*Reading Guide* xv

1 Introduction 1
1.1 Statistical Numerical Approximation 1
1.2 The Game Theoretic Perspective 4
1.3 In the Setting of Sobolev Spaces 7
1.4 Uncertainty Quantification and Probabilistic Numerics 19
1.5 Structure of the Book 20

**Part I The Sobolev Space Setting** 23

2 Sobolev Space Basics 25
2.1 The Sobolev Space 25
2.2 The Operator and Its Corresponding Energy Norm 27

3 Optimal Recovery Splines 34
3.1 Information-Based Complexity 34
3.2 Optimal Recovery 35
3.3 Variational Properties of Optimal Recovery Splines 36

4 Numerical Homogenization 38
4.1 A Short Review of Classical Homogenization 38
4.2 The Numerical Homogenization Problem 47
4.3 Indicator and Dirac Delta Functions as $\phi_i$ 51
4.4 Accuracy 54
4.5 Exponential Decay 54
4.6 Local Polynomials as $\phi_{i,\alpha}$ 58



© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-108-48436-7 — Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization
Houman Owhadi , Clint Scovel
Table of Contents
More Information



4.7 A Short Review of the Localization Problem 59
4.8 A Short Review of Optimal Recovery Splines in Numerical Analysis 61

5 Operator-Adapted Wavelets 63
5.1 A Short Review 63
5.2 Overview of the Construction of Operator-Adapted Wavelets 65
5.3 Non-adapted Prewavelets as $\phi_i^{(k)}$ 66
5.4 Operator-Adapted Prewavelets 73
5.5 Multiresolution Decomposition of $\mathcal{H}_0^s(\Omega)$ 74
5.6 Operator-Adapted Wavelets 76
5.7 Uniformly Bounded Condition Numbers 79
5.8 Multiresolution Decomposition of $u \in H_0^s(\Omega)$ 81
5.9 Interpolation Matrix $R^{(k-1,k)}$ 84
5.10 The Discrete Gamblet Decomposition 86
5.11 Local Polynomials as $\phi_i^{(k)}$ 88

6 Fast Solvers 90
6.1 A Short Review 90
6.2 The Gamblet Transform and Solve 92
6.3 Sparse and Rank-Revealing Representation of the Green's Function 94
6.4 Numerical Illustrations of the Gamblet Transform and Solve 95
6.5 The Fast Gamblet Transform 99

**Part II The Game Theoretic Approach** 103

7 Gaussian Fields 105
7.1 Gaussian Random Variable 105
7.2 Gaussian Random Vector 106
7.3 Gaussian Space 108
7.4 Conditional Covariance and Precision Matrix 109
7.5 Gaussian Process 112
7.6 Gaussian Measure on a Hilbert Space 113
7.7 Gaussian Field on a Hilbert Space 115
7.8 Canonical Gaussian Field on $(\mathcal{H}_0^s(\Omega), \|\cdot\|)$ in Dual Pairing with $(\mathcal{H}^{-s}(\Omega), \|\cdot\|_*)$ 116
7.9 Degenerate Noncentered Gaussian Fields on $\mathcal{H}_0^s(\Omega)$ in Dual Pairing with $\mathcal{H}^{-s}(\Omega)$ 118

8 Optimal Recovery Games on $\mathcal{H}_0^s(\Omega)$ 119
8.1 A Simple Finite Game 119
8.2 A Simple Optimal Recovery Game on $\mathbb{R}^n$ 122

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-108-48436-7 — Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization
Houman Owhadi , Clint Scovel
Table of Contents
More Information



8.3 An Optimal Recovery Game on $\mathcal{H}_0^s(\Omega)$ 124
8.4 Randomized Strategies 124
8.5 Optimal Mixed Strategies 126

9 Gamblets 131
9.1 Elementary Gambles/Bets 131
9.2 Conditional Distribution of the Gaussian Field 133
9.3 Screening Effect 134

10 Hierarchical Games 137
10.1 Introduction 137
10.2 Downscaling Game 139
10.3 The Sequence of Approximations Is a Martingale 142
10.4 Sparse Representation of Gaussian Fields 144
10.5 Probabilistic Interpretation of Numerical Errors 145
10.6 Upscaling with Nested Games 146

**Part III The Banach Space Setting** 149

11 Banach Space Basics 151

12 Optimal Recovery Splines 154
12.1 Projection Properties 154
12.2 Optimal Recovery 156
12.3 Variational Properties 158
12.4 Duality 158

13 Gamblets 160
13.1 Prewavelets 160
13.2 Multiresolution Decomposition of $\mathcal{B}$ 162
13.3 Operator-Adapted Wavelets 163
13.4 Dual Wavelets 165
13.5 Multiresolution Decomposition of $u \in \mathcal{B}$ 168
13.6 Interpolation Matrices 170
13.7 The Gamblet Transform and Gamblet Decomposition 172
13.8 Multiresolution Representation of $Q$ 174
13.9 The Schur Complement $\Theta^{(k)}/\Theta^{(k-1)}$ and $B^{(k)}$ 174
13.10 Geometry of Gamblets 180
13.11 Table of Gamblet Identities 193

14 Bounded Condition Numbers 195
14.1 Notation and Structure Constants 195
14.2 Bounds on $A^{(k)}$ 196
14.3 Bounds on $B^{(k)}$ 196

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-108-48436-7 — Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization
Houman Owhadi , Clint Scovel
Table of Contents
More Information



14.4 Bounds on $N^{(k),T}N^{(k)}$ 198
14.5 Alternate Bounding Mechanism for $B^{(k)}$ 202
14.6 Stability Conditions 204
14.7 Minimum Angle between Gamblets 206
14.8 Sobolev Spaces 208
14.9 Useful Properties of the Structure Constants 250

15 Exponential Decay 252
15.1 Introduction 252
15.2 Subspace Decomposition 253
15.3 Frame Inequalities in Dual Norms 264
15.4 Sobolev Spaces 269

16 Fast Gamblet Transform 297
16.1 Hierarchy of Distances 297
16.2 Hierarchy of Localized Gamblets 302
16.3 The Fast Gamblet Transform and Gamblet Decomposition 305
16.4 Accuracy vs. Complexity Estimates 310
16.5 Sobolev Spaces 341

**Part IV Game Theoretic Approach on Banach Spaces** 345

17 Gaussian Measures, Cylinder Measures, and Fields on $\mathcal{B}$ 347
17.1 Gaussian Measure 347
17.2 Gaussian Field 349
17.3 Gaussian Field and Duality Pairing 350
17.4 Weak Distributions and Cylinder Measures 351
17.5 Gaussian Cylinder Measures as Weak Limits of Gaussian Measures 353
17.6 Canonical Gaussian Field 353
17.7 Canonical Construction 354
17.8 Conditional Expectation and Covariance 355
17.9 When $\mathcal{B} = \mathbb{R}^n$ 358

18 Optimal Recovery Games on $\mathcal{B}$ 360
18.1 Optimal Recovery Game 360
18.2 Optimal Strategies 363

19 Game Theoretic Interpretation of Gamblets 370
19.1 With Two Scales 370
19.2 With Multiple Scales 371
19.3 Conditional Covariances 373
19.4 Sparse Representation of Gaussian Processes 375
19.5 Table of Gaussian Process Regression Identities 376

20 Survey of Statistical Numerical Approximation 378

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-108-48436-7 — Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization
Houman Owhadi , Clint Scovel
Table of Contents
More Information



**Part V Applications, Developments, and Open Problems** 387

21 Positive Definite Matrices 389
21.1 The Setting 389
21.2 The Hierarchy of Labels and Measurement Matrices 389
21.3 The Gamblet Transform and Gamblet Decomposition 390
21.4 Multiresolution Decomposition of $A^{-1}$ 393
21.5 Bounded Condition Numbers 395
21.6 Exponential Decay 401
21.7 The Fast Gamblet Transform on $\mathbb{R}^N$ 404
21.8 On Universality 405

22 Nonsymmetric Operators 406
22.1 Example: Nondivergence Form Operators 407
22.2 Example: Symmetrization with the Inverse Laplacian 408

23 Time-Dependent Operators 410
23.1 Scalar-Wave PDEs 410
23.2 Parabolic PDEs 419

24 Dense Kernel Matrices 421
24.1 The Problem 421
24.2 The Algorithm 422
24.3 Why Does It Work? 424

**Part VI Appendix** 427

25 Fundamental Concepts 429
25.1 Spaces and Mappings 429
25.2 Banach and Hilbert Spaces 431
25.3 The Euclidean Space $\mathbb{R}^N$ 436
25.4 Measure and Integration 438
25.5 Random Variables 440
25.6 Reproducing Kernel Hilbert Spaces 443

*Bibliography* 444
*Algorithms* 460
*Glossary* 461
*Nomenclature* 463
*Index* 467
*Identities* 471

© in this web service Cambridge University Press www.cambridge.org