

OPERATOR-ADAPTED WAVELETS, FAST SOLVERS, AND NUMERICAL HOMOGENIZATION

Although numerical approximation and statistical inference are traditionally covered as entirely separate subjects, they are intimately connected through the common purpose of making estimations with partial information. This book explores these connections from a game and decision theoretic perspective, showing how they constitute a pathway to developing simple and general methods for solving fundamental problems in both areas. It illustrates these interplays by addressing problems related to numerical homogenization, operator-adapted wavelets, fast solvers, and Gaussian processes. This perspective reveals much of their essential anatomy and greatly facilitates advances in these areas, thereby appearing to establish a general principle for guiding the process of scientific discovery. This book is designed for graduate students, researchers, and engineers in mathematics, applied mathematics, and computer science, and particularly researchers interested in drawing on and developing this interface among approximation, inference, and learning.

HOUMAN OWHADI is Professor of Applied and Computational Mathematics and Control and Dynamical Systems in the Department of Computing and Mathematical Sciences of the California Institute of Technology. He earned an M.Sc. from the École Polytechnique in 1994 and was a high civil servant in the Corps des Ponts et Chaussées until 2001. He earned his Ph.D. in probability theory from the École Polytechnique Fédérale de Lausanne in 2001 under the supervision of Gérard Ben Arous and joined the Centre National de la Recherche Scientifique (CNRS) during the same year following a postdoctorate position at Technion. He moved to the California Institute of Technology in 2004. Owhadi serves as an associate editor of the *SIAM Journal on Numerical Analysis*, the *SIAM/ASA Journal on Uncertainty Quantification*, the *International Journal of Uncertainty Quantification*, the *Journal of Computational Dynamics*, and *Foundations of Data Science*. He is one of the main editors of the *Springer Handbook of Uncertainty Quantification*. His research interests concern the exploration of interplays among numerical approximation, statistical inference, and learning from a game theoretic perspective, especially the facilitation/automation possibilities emerging from these interplays. Owhadi was awarded the 2019 Germund Dahlquist Prize by the Society for Industrial and Applied Mathematics.

CLINT SCOVEL is Research Associate in the Computing and Mathematical Sciences Department at the California Institute of Technology, after a 26-year career at Los Alamos National Laboratory, including foundational research in symplectic algorithms and machine learning. He received his Ph.D. in mathematics from the Courant Institute of Mathematics at New York University in 1983 under the supervision of Henry McKean.

The *Cambridge Monographs on Applied and Computational Mathematics* series reflects the crucial role of mathematical and computational techniques in contemporary science. The series publishes expositions on all aspects of applicable and numerical mathematics, with an emphasis on new developments in this fast-moving area of research.

State-of-the-art methods and algorithms as well as modern mathematical descriptions of physical and mechanical ideas are presented in a manner suited to graduate research students and professionals alike. Sound pedagogical presentation is a prerequisite. It is intended that books in the series will serve to inform a new generation of researchers.

A complete list of books in the series can be found at
www.cambridge.org/mathematics.

Recent titles include the following:

19. Matrix preconditioning techniques and applications, *Ke Chen*
20. Greedy approximation, *Vladimir Temlyakov*
21. Spectral methods for time-dependent problems, *Jan Hesthaven, Sigal Gottlieb & David Gottlieb*
22. The mathematical foundations of mixing, *Rob Sturman, Julio M. Ottino & Stephen Wiggins*
23. Curve and surface reconstruction, *Tamal K. Dey*
24. Learning theory, *Felipe Cucker & Ding Xuan Zhou*
25. Algebraic geometry and statistical learning theory, *Sumio Watanabe*
26. A practical guide to the invariant calculus, *Elizabeth Louise Mansfield*
27. Difference equations by differential equation methods, *Peter E. Hydon*
28. Multiscale methods for Fredholm integral equations, *Zhongying Chen, Charles A. Micchelli & Yuesheng Xu*
29. Partial differential equation methods for image inpainting, *Carola-Bibiane Schönlieb*
30. Volterra integral equations, *Hermann Brunner*
31. Symmetry, phase modulation and nonlinear waves, *Thomas J. Bridges*
32. Multivariate approximation, *Vladimir Temlyakov*
33. Mathematical modelling of the human cardiovascular system, *Alfio Quarteroni, Luca Dede', Andrea Manzoni & Christian Vergara*
34. Numerical bifurcation analysis of maps, *Yuri A. Kuznetsov & Hil G.E. Meijer*
35. Operator-adapted wavelets, fast solvers, and numerical homogenization, *Houman Owhadi & Clint Scovel*

Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization

From a Game Theoretic Approach to Numerical Approximation
and Algorithm Design

HOUMAN OWHADI
California Institute of Technology

CLINT SCOVEL
California Institute of Technology



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
978-1-108-48436-7 — Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization
Houman Owhadi, Clint Scovel
Frontmatter
[More Information](#)

CAMBRIDGE UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India
79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781108484367

DOI: 10.1017/9781108594967

© Houman Owhadi and Clint Scovel 2019

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2019

Printed in the United Kingdom by TJ International Ltd., Padstow, Cornwall

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data

Names: Owhadi, Houman, author. | Scovel, Clint, 1955– author.

Title: Operator-adapted wavelets, fast solvers, and numerical homogenization : from a game theoretic approach to numerical approximation and algorithm design / Houman Owhadi (California Institute of Technology),

Clint Scovel (California Institute of Technology).

Description: Cambridge; New York, NY: Cambridge University Press, 2019. |

Series: Cambridge monographs on applied and computational mathematics; 35

Identifiers: LCCN 2019007312 | ISBN 9781108484367 (hardback)

Subjects: LCSH: Approximation theory. | Estimation theory. | Mathematical statistics.

Classification: LCC QA221 .O94 2019 | DDC 515–dc23

LC record available at <https://lccn.loc.gov/2019007312>

ISBN 978-1-108-48436-7 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Cambridge University Press
978-1-108-48436-7 — Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization
Houman Owhadi , Clint Scovel
Frontmatter
[More Information](#)

For Areen, Julien, Kailo, and Ané

Contents

	<i>Preface</i>	page xiii
	<i>Acknowledgments</i>	xiv
	<i>Reading Guide</i>	xv
1	Introduction	1
	1.1 Statistical Numerical Approximation	1
	1.2 The Game Theoretic Perspective	4
	1.3 In the Setting of Sobolev Spaces	7
	1.4 Uncertainty Quantification and Probabilistic Numerics	19
	1.5 Structure of the Book	20
	Part I The Sobolev Space Setting	23
2	Sobolev Space Basics	25
	2.1 The Sobolev Space	25
	2.2 The Operator and Its Corresponding Energy Norm	27
3	Optimal Recovery Splines	34
	3.1 Information-Based Complexity	34
	3.2 Optimal Recovery	35
	3.3 Variational Properties of Optimal Recovery Splines	36
4	Numerical Homogenization	38
	4.1 A Short Review of Classical Homogenization	38
	4.2 The Numerical Homogenization Problem	47
	4.3 Indicator and Dirac Delta Functions as ϕ_i	51
	4.4 Accuracy	54
	4.5 Exponential Decay	54
	4.6 Local Polynomials as $\phi_{i,\alpha}$	58

viii	<i>Contents</i>	
4.7	A Short Review of the Localization Problem	59
4.8	A Short Review of Optimal Recovery Splines in Numerical Analysis	61
5	Operator-Adapted Wavelets	63
5.1	A Short Review	63
5.2	Overview of the Construction of Operator-Adapted Wavelets	65
5.3	Non-adapted Prewavelets as $\phi_i^{(k)}$	66
5.4	Operator-Adapted Prewavelets	73
5.5	Multiresolution Decomposition of $\mathcal{H}_0^s(\Omega)$	74
5.6	Operator-Adapted Wavelets	76
5.7	Uniformly Bounded Condition Numbers	79
5.8	Multiresolution Decomposition of $u \in H_0^s(\Omega)$	81
5.9	Interpolation Matrix $R^{(k-1,k)}$	84
5.10	The Discrete Gamblet Decomposition	86
5.11	Local Polynomials as $\phi_i^{(k)}$	88
6	Fast Solvers	90
6.1	A Short Review	90
6.2	The Gamblet Transform and Solve	92
6.3	Sparse and Rank-Revealing Representation of the Green's Function	94
6.4	Numerical Illustrations of the Gamblet Transform and Solve	95
6.5	The Fast Gamblet Transform	99
	Part II The Game Theoretic Approach	103
7	Gaussian Fields	105
7.1	Gaussian Random Variable	105
7.2	Gaussian Random Vector	106
7.3	Gaussian Space	108
7.4	Conditional Covariance and Precision Matrix	109
7.5	Gaussian Process	112
7.6	Gaussian Measure on a Hilbert Space	113
7.7	Gaussian Field on a Hilbert Space	115
7.8	Canonical Gaussian Field on $(\mathcal{H}_0^s(\Omega), \ \cdot\)$ in Dual Pairing with $(\mathcal{H}^{-s}(\Omega), \ \cdot\ _*)$	116
7.9	Degenerate Noncentered Gaussian Fields on $\mathcal{H}_0^s(\Omega)$ in Dual Pairing with $\mathcal{H}^{-s}(\Omega)$	118
8	Optimal Recovery Games on $\mathcal{H}_0^s(\Omega)$	119
8.1	A Simple Finite Game	119
8.2	A Simple Optimal Recovery Game on \mathbb{R}^n	122

<i>Contents</i>		ix
8.3	An Optimal Recovery Game on $\mathcal{H}_0^s(\Omega)$	124
8.4	Randomized Strategies	124
8.5	Optimal Mixed Strategies	126
9	Gamblets	131
9.1	Elementary Gambles/Bets	131
9.2	Conditional Distribution of the Gaussian Field	133
9.3	Screening Effect	134
10	Hierarchical Games	137
10.1	Introduction	137
10.2	Downscaling Game	139
10.3	The Sequence of Approximations Is a Martingale	142
10.4	Sparse Representation of Gaussian Fields	144
10.5	Probabilistic Interpretation of Numerical Errors	145
10.6	Upscaling with Nested Games	146
Part III The Banach Space Setting		149
11	Banach Space Basics	151
12	Optimal Recovery Splines	154
12.1	Projection Properties	154
12.2	Optimal Recovery	156
12.3	Variational Properties	158
12.4	Duality	158
13	Gamblets	160
13.1	Prewavelets	160
13.2	Multiresolution Decomposition of \mathcal{B}	162
13.3	Operator-Adapted Wavelets	163
13.4	Dual Wavelets	165
13.5	Multiresolution Decomposition of $u \in \mathcal{B}$	168
13.6	Interpolation Matrices	170
13.7	The Gamble Transform and Gamble Decomposition	172
13.8	Multiresolution Representation of Q	174
13.9	The Schur Complement $\Theta^{(k)}/\Theta^{(k-1)}$ and $B^{(k)}$	174
13.10	Geometry of Gamblets	180
13.11	Table of Gamble Identities	193
14	Bounded Condition Numbers	195
14.1	Notation and Structure Constants	195
14.2	Bounds on $A^{(k)}$	196
14.3	Bounds on $B^{(k)}$	196

x	<i>Contents</i>	
14.4	Bounds on $N^{(k),T} N^{(k)}$	198
14.5	Alternate Bounding Mechanism for $B^{(k)}$	202
14.6	Stability Conditions	204
14.7	Minimum Angle between Gamblets	206
14.8	Sobolev Spaces	208
14.9	Useful Properties of the Structure Constants	250
15	Exponential Decay	252
15.1	Introduction	252
15.2	Subspace Decomposition	253
15.3	Frame Inequalities in Dual Norms	264
15.4	Sobolev Spaces	269
16	Fast Gamblet Transform	297
16.1	Hierarchy of Distances	297
16.2	Hierarchy of Localized Gamblets	302
16.3	The Fast Gamblet Transform and Gamblet Decomposition	305
16.4	Accuracy vs. Complexity Estimates	310
16.5	Sobolev Spaces	341
	Part IV Game Theoretic Approach on Banach Spaces	345
17	Gaussian Measures, Cylinder Measures, and Fields on \mathcal{B}	347
17.1	Gaussian Measure	347
17.2	Gaussian Field	349
17.3	Gaussian Field and Duality Pairing	350
17.4	Weak Distributions and Cylinder Measures	351
17.5	Gaussian Cylinder Measures as Weak Limits of Gaussian Measures	353
17.6	Canonical Gaussian Field	353
17.7	Canonical Construction	354
17.8	Conditional Expectation and Covariance	355
17.9	When $\mathcal{B} = \mathbb{R}^n$	358
18	Optimal Recovery Games on \mathcal{B}	360
18.1	Optimal Recovery Game	360
18.2	Optimal Strategies	363
19	Game Theoretic Interpretation of Gamblets	370
19.1	With Two Scales	370
19.2	With Multiple Scales	371
19.3	Conditional Covariances	373
19.4	Sparse Representation of Gaussian Processes	375
19.5	Table of Gaussian Process Regression Identities	376
20	Survey of Statistical Numerical Approximation	378

Contents

xi

	Part V Applications, Developments, and Open Problems	387
21	Positive Definite Matrices	389
21.1	The Setting	389
21.2	The Hierarchy of Labels and Measurement Matrices	389
21.3	The Gamblet Transform and Gamblet Decomposition	390
21.4	Multiresolution Decomposition of A^{-1}	393
21.5	Bounded Condition Numbers	395
21.6	Exponential Decay	401
21.7	The Fast Gamblet Transform on \mathbb{R}^N	404
21.8	On Universality	405
22	Nonsymmetric Operators	406
22.1	Example: Nondivergence Form Operators	407
22.2	Example: Symmetrization with the Inverse Laplacian	408
23	Time-Dependent Operators	410
23.1	Scalar-Wave PDEs	410
23.2	Parabolic PDEs	419
24	Dense Kernel Matrices	421
24.1	The Problem	421
24.2	The Algorithm	422
24.3	Why Does It Work?	424
	Part VI Appendix	427
25	Fundamental Concepts	429
25.1	Spaces and Mappings	429
25.2	Banach and Hilbert Spaces	431
25.3	The Euclidean Space \mathbb{R}^N	436
25.4	Measure and Integration	438
25.5	Random Variables	440
25.6	Reproducing Kernel Hilbert Spaces	443
	<i>Bibliography</i>	444
	<i>Algorithms</i>	460
	<i>Glossary</i>	461
	<i>Nomenclature</i>	463
	<i>Index</i>	467
	<i>Identities</i>	471

Preface

Although numerical approximation and statistical inference are traditionally covered as entirely separate subjects, they are intimately connected through the common purpose of making estimations with partial information. This shared purpose is currently stimulating a growing interest in statistical inference/machine learning approaches to solving partial differential equations (PDEs) [238, 259], in the use of randomized algorithms in linear algebra [153], and in the merging of numerical errors with modeling errors in uncertainty quantification [158].

While this interest might be perceived as a recent phenomenon, interplays between numerical approximation and statistical inference are not new. Indeed, they can be traced back to Poincaré’s course in probability theory [257] and to the pioneering investigations of Sul’din [296], Palasti and Renyi [253], Sard [266], Kimeldorf and Wahba [180] (on the correspondence between Bayesian estimation and spline smoothing/interpolation [312]), and Larkin [195] (on the correspondence between Gaussian process regression and numerical approximation). Although their study initially “attracted little attention among numerical analysts” [195], it was revived in information-based complexity (IBC) [306], Bayesian numerical analysis [95], and more recently probabilistic numerics [158].

This book is an invitation to explore these connections from the consolidating perspective of game/decision theory. It is motivated by the suggestion that these confluences might not just be objects of curiosity but constitute a pathway to developing simple and general methods for solving fundamental problems in both areas. The resulting methods presented in this book are related to numerical homogenization, operator-adapted wavelets, fast solvers, and Gaussian processes.

Acknowledgments

The authors gratefully acknowledge support of this work by the Air Force Office of Scientific Research and the Defense Advanced Research Projects Agency (DARPA) Enabling Quantification of Uncertainty in Physical Systems (EQUiPS) program under award number FA9550-16-1-0054 (Computational Information Games), and the Air Force Office of Scientific Research under award number FA9550-18-1-0271 (Games for Computation and Learning). The authors also thank Max Budninskiy, Jean-Luc Cambier, Mathieu Desbrun, Liu Diyi, Karthik Duraisamy, Fariba Fahroo, Naomi Feldheim, Reza Malek-Madani, George Papanicolaou, Florian Schäfer, Peter Schröder, Bruce Suter, Joel Tropp, Gene Ryan Yoo, Ofer Zeitouni, and Lei Zhang for comments on the technical report [242] and earlier draft versions of the book. A special thanks goes to Don Hush for a thorough reading of the final draft along with many useful comments and suggestions.

Although the main content of this book is based on the technical report [242], to make it a comprehensive treatment, several elements of Schäfer, Sullivan, and Owhadi [270] have also been included. These include, in Section 14.6, an expanded version of [270, Lem. 3.40] and the further development of the relaxations of [242, Conds. 2.13 and 9.8] obtained in [270, Thm. 9.3], which result in the improvement Theorem 14.13 of [242, Thm. 10.9]. These relaxations enable the proof that the gamblets associated with hierarchies of measurement functions consisting of masses of Dirac or Haar prewavelets produce uniformly bounded condition numbers for the fundamental matrix inversions of the Gamblet Transform algorithm. We also thank Florian Schäfer for pointing out an overestimation in our complexity bounds, resulting in an improvement of the exponent from $3d$ to $2d + 1$ in Theorem 16.36.

Reading Guide

This book has two main objectives. One is to explore interplays between numerical approximation and statistical inference using game/decision theory as a consolidating perspective. Another is to illustrate how these connections can be used to derive simple and general methods for solving fundamental problems in both areas.

Since the numerical approximation methods can also be presented in the deterministic setting of optimal recovery without any a priori knowledge of probability theory or statistics, this book has been structured in a manner that would allow a reader to understand them by reading the optimal recovery portions of the book, Parts I and III, without having to read their game/decision theoretic origin/interpretation in Parts II and IV. Although Parts II and IV incorporate self-contained chapters on Gaussian processes and Gaussian fields, they also cover the game/decision theoretic origins/interpretations of the numerical approximation methods of Parts I and III and present what can be learned about Gaussian process regression from these methods.

The introduction that follows, on the other hand, has been written in the spirit of narrating a unified story and describes the content of this book from a close combination of both classical and statistical perspectives on numerical approximation.

To assist in making the text as self-contained as possible, “Fundamental Concepts,” “Nomenclature,” and “Glossary” sections have been included in the Appendix.