OPERATOR-ADAPTED WAVELETS, FAST SOLVERS, AND NUMERICAL HOMOGENIZATION

Although numerical approximation and statistical inference are traditionally covered as entirely separate subjects, they are intimately connected through the common purpose of making estimations with partial information. This book explores these connections from a game and decision theoretic perspective, showing how they constitute a pathway to developing simple and general methods for solving fundamental problems in both areas. It illustrates these interplays by addressing problems related to numerical homogenization, operator-adapted wavelets, fast solvers, and Gaussian processes. This perspective reveals much of their essential anatomy and greatly facilitates advances in these areas, thereby appearing to establish a general principle for guiding the process of scientific discovery. This book is designed for graduate students, researchers, and engineers in mathematics, applied mathematics, and computer science, and particularly researchers interested in drawing on and developing this interface among approximation, inference, and learning.

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Operator-Adapted Wavelets, Fast Solvers, and Numerical Homogenization

From a Game Theoretic Approach to Numerical Approximation and Algorithm Design

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For Areen, Julien, Kailo, and Ané

Contents

Preface

Although numerical approximation and statistical inference are traditionally covered as entirely separate subjects, they are intimately connected through the common purpose of making estimations with partial information. This shared purpose is currently stimulating a growing interest in statistical inference/machine learning approaches to solving partial differential equations (PDEs) [238, 259], in the use of randomized algorithms in linear algebra [153], and in the merging of numerical errors with modeling errors in uncertainty quantification [158].

While this interest might be perceived as a recent phenomenon, interplays between numerical approximation and statistical inference are not new. Indeed, they can be traced back to Poincaré's course in probability theory [257] and to the pioneering investigations of Sul'din [296], Palasti and Renyi [253], Sard [266], Kimeldorf and Wahba [180] (on the correspondence between Bayesian estimation and spline smoothing/interpolation [312]), and Larkin [195] (on the correspondence between Gaussian process regression and numerical approximation). Although their study initially "attracted little attention among numerical analysts" [195], it was revived in information-based complexity (IBC) [306], Bayesian numerical analysis [95], and more recently probabilistic numerics [158].

This book is an invitation to explore these connections from the consolidating perspective of game/decision theory. It is motivated by the suggestion that these confluences might not just be objects of curiosity but constitute a pathway to developing simple and general methods for solving fundamental problems in both areas. The resulting methods presented in this book are related to numerical homogenization, operator-adapted wavelets, fast solvers, and Gaussian processes.

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Although the main content of this book is based on the technical report [242], to make it a comprehensive treatment, several elements of Schäfer, Sullivan, and Owhadi [270] have also been included. These include, in Section 14.6, an expanded version of [270, Lem. 3.40] and the further development of the relaxations of [242, Conds. 2.13 and 9.8] obtained in [270, Thm. 9.3], which result in the improvement Theorem 14.13 of [242, Thm. 10.9] . These relaxations enable the proof that the gamblets associated with hierarchies of measurement functions consisting of masses of Dirac or Haar prewavelets produce uniformly bounded condition numbers for the fundamental matrix inversions of the Gamblet Transform algorithm. We also thank Florian Schäfer for pointing out an overestimation in our complexity bounds, resulting in an improvement of the exponent from $3d$ to $2d + 1$ in Theorem 16.36.

Reading Guide

This book has two main objectives. One is to explore interplays between numerical approximation and statistical inference using game/decision theory as a consolidating perspective. Another is to illustrate how these connections can be used to derive simple and general methods for solving fundamental problems in both areas.

Since the numerical approximation methods can also be presented in the deterministic setting of optimal recovery without any a priori knowledge of probability theory or statistics, this book has been structured in a manner that would allow a reader to understand them by reading the optimal recovery portions of the book, Parts I and III, without having to read their game/decision theoretic origin/interpretation in Parts II and IV. Although Parts II and IV incorporate self-contained chapters on Gaussian processes and Gaussian fields, they also cover the game/decision theoretic origins/interpretations of the numerical approximation methods of Parts I and III and present what can be learned about Gaussian process regression from these methods.

The introduction that follows, on the other hand, has been written in the spirit of narrating a unified story and describes the content of this book from a close combination of both classical and statistical perspectives on numerical approximation.

To assist in making the text as self-contained as possible, "Fundamental Concepts," "Nomenclature," and "Glossary" sections have been included in the Appendix.