PART ONE

TO A_∞ AND BEYOND

1
Categories

In this chapter we review some basics of category theory and representation theory. We will extend the representation theory of algebras to categories and discuss Morita equivalence in this context. Most of the material covered can be found in every standard book about category theory or representation theory [176, 148, 258]. We include it to fix notation and as a reference because in later chapters we will expand this formalism to the world of A_∞-categories.

1.1 Categories

Definition 1.1 A *category* C consists of a collection of objects $\mathrm{Ob}(\mathsf{C})$ and for each pair of objects $A, B \in \mathrm{Ob}(\mathsf{C})$ there is a set $\mathsf{C}(A, B)$. This set is the *hom-space* from A to B and its elements are called *morphisms*. They satisfy the following properties:

- morphisms can be composed: if $\phi \in \mathsf{C}(A, B)$ and $\psi \in \mathsf{C}(B, C)$ then $\psi\phi \in \mathsf{C}(A, C)$,
- the composition is associative: $(\phi\psi)\chi = \phi(\psi\chi)$ whenever it is defined,
- for each object A there is an identity morphism $\mathbb{1}_A \in \mathsf{C}(A, A)$ such that for $\phi \in \mathsf{C}(A, B)$ we have $\phi = \phi\mathbb{1}_A = \mathbb{1}_B\phi$.

A morphism $\phi \in \mathsf{C}(A, B)$ is called an *isomorphism* if there is a $\psi \in \mathsf{C}(B, A)$ such that $\phi\psi = \mathbb{1}_B$ and $\psi\phi = \mathbb{1}_A$. In that case, A and B are called *isomorphic*.

Example 1.2 The standard example of a category is $\mathtt{Sets}$: its objects are sets and morphisms are maps between sets. Two sets are isomorphic if they have the same cardinality.

Example 1.3 (Examples from algebra) For every algebraic structure we can construct a corresponding category; its objects are sets equipped with this algebraic structure and its morphisms are maps preserving this structure. Examples of this construction are $\mathtt{Groups}$, $\mathtt{Fields}$ and $\mathtt{Rings}$.

Two important categories we will often use are $\mathtt{VECT}(\mathbb{k})$ and $\mathtt{vect}(\mathbb{k})$, which are the categories whose objects are all $\mathbb{k}$-vector spaces and all finite-dimensional $\mathbb{k}$-vector spaces and whose morphisms are linear maps.

Example 1.4 (Examples from geometry) In topology the category $\mathtt{Top}$ consists of all topological spaces with continuous maps as morphisms, while $\mathtt{Top}_*$ consists of pairs of a topological space and a point in this space, together with morphisms that map the selected points to each other.

Similarly, in differential geometry we can construct $\mathtt{Man}$ (and $\mathtt{Man}_*$) whose objects are (pointed) manifolds and whose morphisms are smooth maps (that identify the selected points).

Every topological space $\mathbb{X}$ can also be considered as a category $\mathtt{Open}(\mathbb{X})$, whose objects are the open subsets and whose morphisms are the inclusions.

All these examples are categories for which the objects are sets with a special structure and the morphisms are maps between these sets preserving the structure. There are also other interesting categories that do not fall into this class.

Example 1.5 The *fundamental groupoid* of a topological space $\Pi_1(\mathbb{X})$ is the category for which the objects are the points in $\mathbb{X}$ and the morphisms are homotopy classes of paths between points. Note that if $\mathbb{X}$ is connected then $\Pi_1(\mathbb{X})(p, p)$ is by definition equal to the fundamental group $\pi_1(\mathbb{X}, p)$.

1.2 Functors

Definition 1.6 A *covariant functor* $\mathcal{F}: \mathtt{C} \to \mathtt{D}$ consists of maps $\mathcal{F}\colon \mathrm{Ob}(\mathtt{C}) \to \mathrm{Ob}(\mathtt{D})$ and $\mathcal{F}\colon \mathtt{C}(A, B) \to \mathtt{D}(\mathcal{F}(A), \mathcal{F}(B))$ such that $\mathcal{F}(\psi\phi) = \mathcal{F}(\psi)\mathcal{F}(\phi)$ and $\mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}$.

Example 1.7 Examples of covariant functors are *forgetful functors*, which consider the same objects and maps but forget some of the structure: $\mathcal{F}: \mathtt{Groups} \to \mathtt{Sets}\colon (G, *) \mapsto G$ or $\mathcal{F}: \mathtt{Rings} \to \mathtt{Groups}\colon (R, +, \times) \mapsto (R, +)$.

Example 1.8 Other functors create more structure, such as $\mathcal{U}\colon \mathtt{Sets} \to \mathtt{VECT}(\mathbb{k})\colon S \mapsto \langle S \rangle$ which associates to each set a vector space with as basis that set. Because a linear map is determined by the images of the basis, the functor turns each map between generators into a linear map between vector spaces.

The group ring construction is also an example of this: it associates to each group its group ring, consisting of all linear combinations of group elements. This functor turns each group morphism into a ring morphism by linearizing it.

Example 1.9 A more fancy example of a covariant functor is the fundamental group, which goes from pointed topological spaces to groups,

$$\pi_1 \colon \mathtt{Top}_* \to \mathtt{Groups} \colon (X, p) \mapsto \pi_1(X, p).$$

If $f \colon (X, p) \to (Y, q)$ is a continuous map, we set $\pi_1(f) \colon \pi_1(X, p) \to \pi_1(Y, q) \colon [\ell] \mapsto [f\ell]$ where $\ell \colon [0, 1] \to X$ is a loop starting at $p \in X$ and $[\ell]$ is its corresponding element in the fundamental group.

Definition 1.10 A *contravariant functor* $\mathcal{F} \colon \mathtt{C} \to \mathtt{D}$ consists of maps $\mathcal{F} \colon \mathrm{Ob}(\mathtt{C}) \to \mathrm{Ob}(\mathtt{D})$ and $\mathcal{F} \colon \mathtt{C}(A, B) \to \mathtt{D}(\mathcal{F}(B), \mathcal{F}(A))$ such that

$$\mathcal{F}(\psi\phi) = \mathcal{F}(\phi)\mathcal{F}(\psi) \quad \text{and} \quad \mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}.$$

If we define the *opposite category* $\mathtt{C}^{\mathrm{op}}$ to be the category with the same objects but

$$\mathtt{C}^{\mathrm{op}}(V, W) := \mathtt{C}(W, V)$$

and the multiplication reversed, then a contravariant functor from $\mathtt{C} \to \mathtt{D}$ is the same as a covariant functor from $\mathtt{C}^{\mathrm{op}} \to \mathtt{D}$.

Example 1.11 The basic example of a contravariant functor is the dual of a vector space

$$-^* \colon \mathtt{vect}(\Bbbk) \to \mathtt{vect}(\Bbbk) \colon V \mapsto V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk).$$

This functor is contravariant because if $f \colon V \to W$ is a linear map, the corresponding dual map $f^* \colon W^* \to V^* \colon \phi \mapsto \phi \circ f$ goes in the opposite direction.

Example 1.12 Another example is the construction that associates to every manifold its ring of smooth functions

$$C_\infty \colon \mathtt{Man} \to \mathtt{Rings} \colon \mathbb{M} \mapsto C_\infty(\mathbb{M}) := \{f \colon \mathbb{M} \to \mathbb{R} \mid f \text{ is smooth}\}.$$

Definition 1.13 A covariant or contravariant functor $\mathcal{F} \colon \mathtt{C} \to \mathtt{D}$ is called *full*, *faithful* or *fully faithful* if all the maps $\mathcal{F} \colon \mathtt{C}(A, B) \to \mathtt{D}(\mathcal{F}(A), \mathcal{F}(B))$ are surjective, injective or bijective. A fully faithful covariant or contravariant functor is called an *equivalence* or *antiequivalence* if for each $B \in \mathrm{Ob}(\mathtt{D})$ there is an $A \in \mathrm{Ob}(\mathtt{C})$ with $\mathcal{F}(A)$ isomorphic to B. This last property is also called *essentially surjective*.

Example 1.14 Define the category $\mathtt{mat}(\Bbbk)$ as follows:

$$\mathrm{Ob}(\mathtt{mat}(\Bbbk)) := \{0, 1, 2, \ldots\} \quad \text{and} \quad \mathtt{mat}(n, m) := \mathrm{Mat}_{m\times n}(\Bbbk).$$

This category is equivalent to the category of vector spaces through the functor

$$\mathcal{F}: \mathtt{mat}(\Bbbk) \to \mathtt{vect}(\Bbbk): n \mapsto \Bbbk^n,$$
$$\mathcal{F}: \mathtt{mat}(n, m) \to \mathtt{vect}(\Bbbk^n, \Bbbk^m): A \mapsto (\phi: x \mapsto Ax),$$

which is fully faithful because every linear map between $\Bbbk^n$ and $\Bbbk^m$ is represented by a unique matrix. It is also essentially surjective because every finite-dimensional vector space is isomorphic to $\Bbbk^n$ for some n.

The main idea is that equivalent categories describe two types of mathematical objects which behave the same. All constructions between one of these types of objects and their morphisms can be translated to the other setting and vice versa. In our particular situation it means that working with vector spaces and linear maps is the same as working with matrices.

Example 1.15 The functor $-^*$ is an antiequivalence for the category of all finite-dimensional vector spaces $\mathtt{vect}(\Bbbk)$, but not for the category $\mathtt{VECT}(\Bbbk)$ of all vector spaces over $\Bbbk$. This is because the dual of an infinite-dimensional vector space never has a countable basis, so $-^*$ is not essentially surjective.

1.3 Natural Transformations

The final main ingredient of category theory is natural transformations. They can be seen as morphisms between functors.

Definition 1.16 A *natural transformation* $\eta: \mathcal{F} \to \mathcal{G}$ between two covariant functors $\mathcal{F}, \mathcal{G}: \mathtt{C} \to \mathtt{D}$ is a collection of $\mathtt{D}$-morphisms $(\eta_X: \mathcal{F}(X) \to \mathcal{G}(X))_{X\in \mathrm{Ob}\,\mathtt{C}}$ such that for every morphism $f: X \to Y$ in $\mathtt{C}$ we have $\eta_Y \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_X$.

Example 1.17 The standard example of a natural transformation is the one between the identity functor $\mathbb{1}: \mathtt{vect}(\Bbbk) \to \mathtt{vect}(\Bbbk)$ and the double dual of a vector space $-^{**}: \mathtt{vect}(\Bbbk) \to \mathtt{vect}(\Bbbk)$:

$$\eta_X: X \to X^{**}: v \mapsto (\mathrm{ev}_v: X^* \to \Bbbk: \phi \mapsto \phi(v))\,.$$

This is the mathematical way of saying that a finite-dimensional vector space and its double dual are canonically isomorphic: there is a standard

Cambridge University Press
978-1-108-48350-6 — A Gentle Introduction to Homological Mirror Symmetry
Raf Bocklandt
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isomorphism between a vector space and its double dual. This is not the case for the single dual: although a finite-dimensional vector space is isomorphic to its dual, to construct an actual isomorphism we need some extra data (like a basis).

Example 1.18 It is also easy to check that every natural transformation of the identity functor $\mathbb{1}: \mathtt{vect}(\Bbbk) \to \mathtt{vect}(\Bbbk)$ to itself is a global rescaling by a common factor $\lambda \in \Bbbk$:

$$(\nu_\lambda)_X : X \to X : v \mapsto \lambda v.$$

Indeed, if $\nu: \mathbb{1} \to \mathbb{1}$ is a natural transformation then it acts by rescaling on the one-dimensional vector space $\Bbbk$: $\nu_\Bbbk : \Bbbk \to \Bbbk : x \to \lambda x$. If v is an element of a vector space X then there is a linear map $f: \Bbbk \to X: 1 \mapsto v$, so $\nu_X(v) = \nu_X(f(1)) = f(\nu_\Bbbk(1)) = f(\lambda) = \lambda v$.

1.4 Linear Categories

If we fix a field $\Bbbk$ we can consider $\Bbbk$-linear categories.

Definition 1.19 A category $\mathtt{C}$ is called a $\Bbbk$-*linear category* if all hom-spaces are $\Bbbk$-vector spaces and the multiplication is bilinear. A functor between two $\Bbbk$-linear categories is called $\Bbbk$-*linear* if the maps between the hom-spaces are $\Bbbk$-linear. An object X in a $\Bbbk$-linear category $\mathtt{C}$ is called a *zero object* if for all $Y \in \mathrm{Ob}(\mathtt{C})$ we have $\mathtt{C}(X, Y) = \mathtt{C}(Y, X) = 0$.

Unless it is specifically stated otherwise, a functor between two $\Bbbk$-linear categories will always be assumed to be $\Bbbk$-linear. This is not a big restriction because almost all natural functors are $\Bbbk$-linear.

Example 1.20 There are many examples of $\Bbbk$-linear categories: $\mathtt{VECT}(\Bbbk)$, $\mathtt{vect}(\Bbbk)$, $\mathtt{mat}(\Bbbk)$. For any $\Bbbk$-algebra A, we have $\Bbbk$-linear categories

- $\mathtt{MOD}\text{-}A$: the category of all *right A-modules*,
- $\mathtt{Mod}\text{-}A$: the category of all *finitely generated right A-modules*, and
- $\mathtt{mod}\text{-}A$: the category of all *finite-dimensional right A-modules*.

The zero object in these categories is the zero-vector space with trivial A-action. The hom-spaces in these categories are often denoted by $\mathrm{Hom}_A(M, N)$ instead of $\mathtt{MOD}\text{-}A(M, N)$. The same triplet exists for left modules as well: $A\text{-}\mathtt{MOD}$, $A\text{-}\mathtt{Mod}$ and $A\text{-}\mathtt{mod}$.

Example 1.21 Any $\mathbb{k}$-algebra A can also be considered as a $\mathbb{k}$-linear category $\mathtt{A}$ with one object $\bigcirc$ and $\mathtt{A}(\bigcirc,\bigcirc) = A$. On the other hand, it is also possible to consider a category $\mathtt{C}$ with more than one object as an algebra: we can take the direct sum of all hom-spaces

$$C := \bigoplus_{X,Y\in \mathrm{Ob}(\mathtt{C})} \mathtt{C}(X, Y).$$

The product of two morphisms is composition if they are composable and zero otherwise. Note that for each object X, the unit $\mathbb{1}_X$ gives an idempotent in the algebra. If there is a finite number of objects the sum of all these idempotents is the unit in the algebra, but if there are infinitely many objects this will be an algebra without a unit.

Note that these operations are not inverses. If we start with a category with a finite number of objects, turn it into an algebra and then back into a category, this new category is not equivalent to the original. There is however a weaker sense in which they are equivalent. This will be explored in the next sections.

1.5 Modules

Given a $\mathbb{k}$-algebra A, a left module consists of a vector space V and an algebra morphism $\rho\colon A \to \mathrm{Hom}_{\mathbb{k}}(V, V)$. If we consider A as a category $\mathtt{A}$ with one object, this is precisely a covariant functor from $\mathtt{A}$ to $\mathtt{Vect}(\mathbb{k})$. A morphism between two modules is a linear map $f\colon V \to W$ such that for all $a \in A$ we have $f \circ \rho_V(a) = \rho_W(a) \circ f$, in which we recognize a natural transformation between the functors ρ_V and ρ_W.

Definition 1.22 If $\mathtt{C}$ is a ($\mathbb{k}$-linear) category we define its *left module category* $\mathtt{C}\text{-}\mathtt{MOD}$ as the category with as objects the ($\mathbb{k}$-linear) functors $\rho\colon \mathtt{C} \to \mathtt{VECT}(\mathbb{k})$ and as morphisms the natural transformations between them. An object of $\mathtt{C}\text{-}\mathtt{MOD}$ is called a *left* $\mathtt{C}$*-module*. The *right module category* $\mathtt{MOD}\text{-}\mathtt{C}$ is the category of contravariant functors from $\mathtt{C}$ to $\mathtt{VECT}(\mathbb{k})$. The objects are *right* $\mathtt{C}$*-modules*.

Example 1.23 If $\mathtt{C}$ is a $\mathbb{k}$-linear category with a finite number of objects and C is the corresponding algebra then we can turn every classical left C-module M into a functor $\mathcal{F}_M\colon \mathtt{C} \to \mathtt{VECT}(\mathbb{k})$ that maps an object X to the vector space $\mathcal{F}_M := \mathbb{1}_X M$ and every morphism $\phi\colon X \to Y$ to $\mathcal{F}_M(\phi)\colon \mathbb{1}_X M \to \mathbb{1}_Y M\colon m \mapsto \phi m$.

Vice versa, every $\mathtt{C}$-module $\mathcal{F}\colon \mathtt{C} \to \mathtt{VECT}(\mathbb{k})$ can be seen as a classical module of the algebra $C = \bigoplus_{X,Y\in \mathrm{Ob}\,\mathtt{C}} \mathtt{C}(X, Y)$ by taking the direct sum $M_{\mathcal{F}} := \bigoplus_{X\in \mathrm{Ob}\,\mathtt{C}} \mathcal{F}(X)$ and letting $\mathcal{F}(\phi)$ act on the appropriate components.

Example 1.24 Many classical constructions in geometry are actually modules of categories.

- Let $C_\infty(\mathbb{M})$ denote the space of smooth functions on a manifold $\mathbb{M}$. If we forget the ring structure on $C_\infty(\mathbb{M})$, the functor $C_\infty\colon \mathtt{Man} \to \mathtt{VECT}(\mathbb{R})$ can be seen as a right module of $\mathtt{Man}$. Similarly, the differential n-forms Ω^n and the de Rham cohomology H^n_{dR} can also be seen as right modules of $\mathtt{Man}$.
- If $\mathscr{V}$ is a vector bundle over a topological space $\mathbb{X}$ then the section functor $\Gamma(\mathscr{V}, -)$, which maps every open $\mathbb{U}$ to its space of sections $\Gamma(\mathscr{V}, \mathbb{U})$, is a contravariant functor from $\mathtt{Open}(\mathbb{X})$ to $\mathtt{VECT}(\mathbb{R})$ and hence a right module of $\mathtt{Open}(\mathbb{X})$. In general, a right module of $\mathtt{Open}(\mathbb{X})$ is also called a *presheaf* of $\mathbb{X}$.
- A *local system* on a topological space $\mathbb{X}$ can be seen as a module of the fundamental groupoid $\Pi_1(\mathbb{X})$: it assigns to each point in $\mathbb{X}$ a vector space and to each path a linear map that only depends on the homotopy class. In other words, it is a vector bundle with a flat connection.

Modules of categories behave in all respects like modules of rings and all familiar concepts hold. We say that a morphism between $\mathtt{C}$-modules is *injective*, *surjective* or *bijective* if its natural transformation is respectively injective, surjective or bijective in every object of $\mathtt{C}$.

For every morphism we can also define its *kernel* and *cokernel* by looking at the kernel and cokernel of the natural transformation in every object. Other notions such as short exact sequences and direct sums and summands can also be defined objectwise.

Example 1.25 The exterior derivative can be viewed as a $\mathtt{Man}$-module morphism $d_n\colon \Omega^n \to \Omega^{n+1}$ and $H^n_{\mathrm{dR}} = \operatorname{Ker} d_n / \operatorname{Im} d_{n-1}$ as right $\mathtt{Man}$-modules. For more details see Section 2.2.3.

Just as we can consider an algebra as a module over itself, we can see every object in a $\mathbb{k}$-linear category as a module of this category: to $X \in \operatorname{Ob}(\mathtt{C})$ we associate the contravariant functor $\mathtt{C}(-, X)$. A morphism between two objects $\phi\colon X \to Y$ will give a natural transformation $\phi \circ -\colon \mathtt{C}(-, X) \to \mathtt{C}(-, Y)\colon f \mapsto \phi \circ f$. Therefore we have a covariant functor $\mathcal{Y}\colon \mathtt{C} \to \mathtt{MOD}\text{-}\mathtt{C}$. This functor is called the *Yoneda embedding*.

Lemma 1.26 (Yoneda lemma) *The functor $\mathcal{Y}\colon \mathtt{C} \to \mathtt{MOD}\text{-}\mathtt{C}$ is fully faithful.*

Yoneda's lemma implies that $\mathtt{C}$ can be seen as a full subcategory of $\mathtt{MOD}\text{-}\mathtt{C}$. In a similar way $\mathtt{C}^{\mathrm{op}}$ embeds in the left module category $\mathtt{C}\text{-}\mathtt{MOD}$.

1.6 Morita Equivalence

Sometimes nonisomorphic algebras can have the equivalent module categories. This phenomenon is called *Morita equivalence* and has been well studied [9, 151].

Example 1.27 If A is an algebra we say that $e \in A$ is an *idempotent* if $e^2 = e$ and it is a *full idempotent* if additionally $AeA = A$. For a full idempotent e we can construct an algebra $B = eAe = \mathrm{Hom}_A(eA, eA)$ and a functor

$$\mathcal{E}\colon \mathtt{MOD}\text{-}A \to \mathtt{MOD}\text{-}B\colon M \mapsto Me \quad \text{with } \mathcal{E}(f\colon M \to N) = f|_{Me}.$$

This functor is fully faithful because we can reconstruct f from $f|_{Me}$ as $M = MAeA = (Me)A$. It is also essentially surjective because

$$\mathcal{E}(N \otimes_B eA) = (N \otimes_B eA)e = N \otimes_B eAe = N \otimes_B B \cong N.$$

Therefore, $\mathcal{E}$ is an equivalence between $\mathtt{MOD}\text{-}A$ and $\mathtt{MOD}\text{-}B$.

Example 1.28 The elementary matrix E_{11} with a 1 in the upper-left corner is a full idempotent for $\mathrm{Mat}_n(A) = \mathrm{Hom}_A(A^{\oplus n}, A^{\oplus n})$, so the latter is Morita equivalent with A.

The previous examples indicate that there are two processes to make Morita equivalences: taking endomorphism rings of certain direct summands (eA) and direct sums ($A^{\oplus n}$). These processes are sufficient:

Theorem 1.29 (Morita [177]) *Two algebras A and B are Morita equivalent if and only if $B \cong e\,\mathrm{Mat}_n(A)e$ for some full idempotent matrix $e \in \mathrm{Mat}_n(A)$.*

These ideas generalize to Morita equivalences for $\mathbb{k}$-linear categories as well.

Definition 1.30 Two categories $\mathtt{C}$ and $\mathtt{D}$ are called *Morita equivalent* if $\mathtt{MOD}\text{-}\mathtt{C}$ and $\mathtt{MOD}\text{-}\mathtt{D}$ are equivalent categories.

Definition 1.31 The *additive completion* $\mathtt{Add}\,\mathtt{C}$ of a $\mathbb{k}$-linear category $\mathtt{C}$ is the category with objects that are finite formal direct sums of objects in $\mathtt{C}$, including the zero sum. We write such objects as $(A_1 \oplus \cdots \oplus A_n)$ with $A_i \in \mathrm{Ob}(\mathtt{C})$ and

$$\mathtt{Add}\,\mathtt{C}(A_1 \oplus \cdots \oplus A_n, B_1 \oplus \cdots \oplus B_m) := \bigoplus_{i,j} \mathtt{C}(A_i, B_j),$$

where the right-hand side is a direct sum of $\mathbb{k}$-vector spaces. The composition of morphisms is the bilinear extension of the composition in $\mathtt{C}$. Note that $\mathtt{C}$ is a full subcategory of $\mathtt{Add}\,\mathtt{C}$.