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Semigroups of Linear Operators
With Applications to Analysis, Probability and Physics

DAVID APPLEBAUM
University of Sheffield



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This book is dedicated to the mathematical community
of Costa Rica. Long may it flourish.

“I hail a semigroup when I see one, and I seem to see them everywhere!”

(Einar Hille, Foreword to [45])

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