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David Barnes , Constanze Roitzheim
Excerpt
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Introduction

Aims and Intended Readership

The aim of this book is to be an accessible introduction to stable homotopy theory that novices, particularly graduate students, can use to learn the fundamentals of the subject. For the experts, we hope to have provided a useful compendium of results across the main areas of stable homotopy theory.

This book is not intended to replace any specific part of the existing literature, but instead to give a smoother, more coherent introduction to stable homotopy theory. We use modern techniques to give a streamlined development that avoids a number of outdated, and often over-complicated, constructions of a suitable stable homotopy category. We cover the most pressing topics for a novice and give a narrative to motivate the development. This narrative is missing from much of the current literature, which often assumes the reader already knows stable homotopy theory and hence understands why any given definition or result is important.

The majority of sections have been written to (hopefully) contain all details required for a graduate student. The remaining sections are intended to give an overview of more specialised or advanced topics, with references to the central texts for those areas. It is hoped that once the reader has read the chapters relevant to their research, they will be well prepared to dive into the rest of the literature and to know what to read next.

Prerequisites

Rather than rewrite many pages on model categories, category theory and unstable homotopy theory, we depend upon several excellent, and quite standard, references. As such, the reader should know a fair amount of point-set topology and algebraic topology. The standard texts are Gray [Gra75], Hatcher [Hat02], May [May99a] and May and Ponto [MP12]. The reader should also know the basics of model categories. The best introductions are Dwyer and Spaliński [DS95] and the first chapters of Hovey [Hov99].

A certain amount of category theory is used throughout, the standard text is Mac Lane [Mac71]. For the chapters on the monoidal smash product, the reader will need some enriched category theory, easily obtained from Kelly [Kel05]. They may also like to have access to Borceaux [Bor94]. The chapter on localisations refers to Hirschhorn [Hir03] for some proofs and technical results, but the reader will not need to have read the book to follow the development.

A Historical Narrative

The book [Jam99] gives a treatment of the history of topology, while the chapter of May [May99b] (50 pages) covers stable homotopy theory from 1945 to 1966. Since then, the pace of development and publication has only quickened, a thorough history of stable homotopy theory would be a book by itself.

A basic problem in homotopy theory is the calculation of the homotopy groups of spheres. This problem is well known to be hopelessly difficult, but certain patterns in the homotopy groups were noticed. The Freudenthal Suspension Theorem gives a clear statement of a major pattern: the group $\pi_n(S^{k+n})$ is independent of n for $n > k + 1$. This and the suspension isomorphisms of homology and cohomology were a starting point of stable homotopy theory.

Calculations continued and the Spanier–Whitehead category was developed to study duality statements. It was also a useful category for the study of spaces under equivalences of stable homotopy groups. However, the Spanier–Whitehead category has some substantial drawbacks, in particular, it does not contain representatives for all reduced cohomology theories.

Several solutions to this were constructed, including Boardman's stable homotopy category, Lima's notion of spectra, Kan's semisimplicial spectra and Whitehead's developments of the notions of spectra. None of these categories were entirely satisfactory, so we jump ahead to Adams's construction of the stable homotopy category [Ada74], which was based on ideas of Boardman. This category contained the Spanier–Whitehead category, represented all cohomology theories and had a commutative smash product.

Having a good construction with sensible axioms allowed for further development of stable homotopy theory. A good notion of "categories of fractions", now known as Bousfield localisations at homology theories, greatly improved the ability to calculate stable homotopy groups via the Adams spectral sequence. Through work of Bousfield, these localisations were further developed

([Bou75], [Bou79]), and vast amounts of calculations were now possible. This led to the introduction of chromatic homotopy theory, which gives a framework for major structural results about the stable homotopy category (see [Rav84] and [Rav92a]), as well as techniques for even further calculations of stable homotopy groups of spheres [Rav86].

The lack of a good commutative monoidal point-set model for the stable homotopy category still held the subject area back. Brown representability posited the existence of function spectra and allowed for some homotopical calculations, but direct constructions were often impossible to give. The study of (commutative) ring spectra up to homotopy was difficult – keeping track of the homotopies and their coherence was particularly burdensome. Moreover, constructions up to homotopy prevented geometric constructions such as bundles of spectra or diagrams of spectra.

The development of coordinate–free spectra by May and others offered several improvements to the area. The use of operads to manage commutative multiplications up to homotopy allowed for serious study of derived algebra in spectra, the so-called “brave new algebra”, see the work of May, Quinn and Ray [May77].

Coordinate–free spectra also led to the development of “spectrification” functors, which simplified the construction of maps between spectra. These functors played a central role in work of Lewis, May and Steinberger [LMSM86], which gave a construction of G-equivariant spectra and the G-equivariant stable homotopy category for G a compact Lie group.

While this technology did allow for a useful definition of an internal function object for spectra, the smash product was still only commutative and associative up to homotopy. The work of Lewis [Lew91] even suggested that there may be no commutative monoidal point-set model for the stable homotopy category, but this pessimism turned out to be unfounded.

Two independent solutions to the problem of commutative smash products came about in surprisingly quick succession: the S–modules of Elmendorff, Kriz, Mandell and May [EKMM97] and the symmetric spectra of Hovey, Shipley and Smith [HSS00]. These references gave closed symmetric monoidal model categories of spectra and model categories of (commutative) ring spectra. By using model categories, one had point-set level smash products and function objects which would have the correct homotopical properties after passing to the homotopy categories.

This reinvigorated the area and allowed for a great deal of further development in “brave new algebra”, namely, the importing of statements from algebra into stable homotopy theory. For example, Hovey, Palmieri and Strickland [HPS97] were able to give an axiomatisation of stable homotopy theories.

Schwede [Sch01a] showed that the model categories of symmetric spectra and S–modules were Quillen equivalent. Yet more symmetric monoidal categories of spectra were constructed by Mandell, May, Schwede and Shipley [MMSS01]. These were all shown to be Quillen equivalent, and a particular highlight is the category of orthogonal spectra. Categories of spectra in categories other than simplicial sets or spaces were given in Schwede [Sch97] and Hovey [Hov01b].

All of these model categories are amenable to the theory of localisations as developed by Hirschhorn [Hir03], giving many point-set models for localisations of spectra and, in particular, those from chromatic homotopy theory.

We are now in the modern era of stable homotopy theory, with current topics such as topological modular forms and its variants, motivic stable homotopy theory, the study of commutative ring spectra and their localisations, Galois extensions of ring spectra and equivariant versions of most of those topics.

Explanation of Contents

We start with a study of stable phenomena, namely, the Freudenthal Suspension Theorem and the suspension isomorphisms of homology and cohomology. We discuss how this leads to the notion of a stable homotopy category and what axioms it should satisfy. We introduce the Spanier–Whitehead category and basic categories of spectra and show how these fail to satisfy those axioms. Using the benefit of hindsight, we then define the stable homotopy category to be the homotopy category of the stable model structure on sequential spectra. This approach avoids the difficulties of extending the Spanier–Whitehead category and the complicated constructions of maps and functions in Adams's category of spectra.

The category of sequential spectra evidently satisfies enough of the axioms to be worth studying further, but it will not be possible to give a category that satisfies all the axioms until after we have introduced the symmetric monoidal categories of symmetric and orthogonal spectra.

We digress from the further development of categories of spectra to ask about a formal framework in which those categories of spectra can be studied. The starting place is a suspension functor on general model categories and how it gives rise to cofibre and fibre sequences, leading to a notion of a stable model category. When the model category is stable, one can extend cofibre and fibre sequences in either direction and prove the fundamental statement: the homotopy category of a stable model category is triangulated. Working in this generality shows clear benefits of stability and simplifies the later work, where we can appeal to the triangulated arguments of these chapters.

The next task is to examine the generalisation of the smash product of the Spanier–Whitehead category to the stable homotopy category. Again, we want an approachable method, so we first show that the homotopy categories of symmetric spectra and orthogonal spectra are equivalent to the stable homotopy category. We then define the smash product and the internal function object of the stable homotopy category as coming from the smash products on orthogonal spectra and symmetric spectra. At this point, we have encountered three models for the stable homotopy category. Each has its own advantages. We have:

Sequential spectra: These are the simplest to define and can be motivated from a discussion of Brown representability. However, they do not have a commutative smash product. The weak equivalences are defined in terms of homotopy groups of spectra, a natural extension of the idea of stable homotopy groups.

Orthogonal spectra: These are slightly more complicated than sequential spectra and can be thought of as sequential spectra with extra structure. Their weak equivalences are defined by the forgetful functor to sequential spectra and, hence, are defined in terms of homotopy groups of spectra. The extra structure allows one to define a symmetric monoidal smash product and an internal function object.

Symmetric spectra: The final model is symmetric spectra (in either pointed topological spaces or pointed simplicial sets). This model is intermediate in its complexity, but the weak equivalences are harder to define. These spectra also have good monoidal properties. Their simplicity allows them to be described as "initial amongst stable model categories" in the sense of Sections 6.8 and 6.9.

The symmetric monoidal versions of spectra lead to the next important topic: spectra with (commutative) ring structures and Spanier–Whitehead duality for spectra, which is essentially a study of duality in the stable homotopy category.

We take the opportunity to consider framings and stable framings. This allows us to construct mapping spaces for an arbitrary model category and mapping spectra for an arbitrary stable model category.

We end the book with a chapter on Bousfield localisation, introducing and motivating the concept and proving a simple existence result for stable model categories. As an application, we discuss p-localisation, p-completion and localisation at complex topological K-theory.

Along the way, we furthermore include important results on stable homotopy theory and suggest further directions. The results include: rigidity and uniqueness of the monoidal structure, a description in terms of modules over spectrally enriched categories, the Adams spectral sequence and chromatic homotopy theory.

An appendix listing the results on model categories that are needed is included for easy reference. Some proofs are given, otherwise clear references are provided.

Omissions

An exhaustive treatment of stable homotopy theory would require several books and be impractical for the needs of many graduate students. Hence, certain topics have been omitted, a list is below. Reasons for the omission vary, from being somewhat outside the scope (stable infinity categories), being a topic that builds upon stable homotopy theory (equivariant or motivic stable homotopy theory), or having good textbooks already, albeit ones that assume a familiarity with stable homotopy theory.

- Infinite loop space machines and operads.
- Right Bousfield localisations.
- Comprehensive treatment of spectral sequences.
- Equivariant stable homotopy theory.
- Motivic stable homotopy theory.
- Stable infinity categories.
- In-depth treatment of K-theory, cobordism and formal group laws.
- The S–modules of Elmendorf, Kriz, Mandell and May.

Convention

Throughout the book we use the convention that the set of natural numbers $\mathbb{N}$ contains 0.

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