

## FORMULATIONS OF GENERAL RELATIVITY

This monograph describes the different formulations of Einstein's General Theory of Relativity. Unlike traditional treatments, Cartan's geometry of fibre bundles and differential forms is placed at the forefront, and a detailed review of the relevant differential geometry is presented. Particular emphasis is given to General Relativity in 4D space-time, in which the concepts of chirality and self-duality begin to play a key role. Associated chiral formulations are catalogued, and shown to lead to many practical simplifications. The book develops the chiral gravitational perturbation theory, in which the spinor formalism plays a central role. The book also presents in detail the twistor description of gravity, as well as its generalisation based on geometry of 3-forms in seven dimensions. Giving valuable insight into the very nature of gravity, this book joins our highly prestigious Cambridge Monographs in Mathematical Physics series. It will interest graduate students and researchers in the fields of Theoretical Physics and Differential Geometry.

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<sup>†</sup> Available in paperback

Cambridge University Press  
978-1-108-48164-9 — Formulations of General Relativity  
Kirill Krasnov  
Frontmatter  
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# Formulations of General Relativity

## Gravity, Spinors and Differential Forms

KIRILL KRASNOV  
*University of Nottingham*



Cambridge University Press  
978-1-108-48164-9 — Formulations of General Relativity  
Kirill Krasnov  
Frontmatter  
[More Information](#)

CAMBRIDGE  
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom  
One Liberty Plaza, 20th Floor, New York, NY 10006, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India  
79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9781108481649](http://www.cambridge.org/9781108481649)  
DOI: 10.1017/9781108674652

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First published 2020

*A catalogue record for this publication is available from the British Library.*

ISBN 978-1-108-48164-9 Hardback

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To the memory of the two Peter Ivanovichs in my life:  
Peter Ivanovich Fomin, who got me interested in gravity,  
and Peter Ivanovich Holod, who formed my taste for mathematics.

Cambridge University Press  
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Cambridge University Press  
978-1-108-48164-9 — Formulations of General Relativity  
Kirill Krasnov  
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## Preface

Give thanks to God, who made necessary things simple, and complicated things unnecessary.

Gregory Skovoroda, *Ukrainian Thinker*, 1722–1794

There is always another way to say the same thing that doesn't look at all like the way you said it before. I don't know what the reason for this is. I think it is somehow a representation of the simplicity of nature? Perhaps a thing is simple if you can describe it fully in several different ways without immediately knowing that you are describing the same thing.

Richard Feynman, *Nobel Lecture*, 1965

Theories of the known, which are described by different physical ideas may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made and hence are not equivalent in the hypotheses one generates from them in ones attempt to understand what is not yet understood. I, therefore, think that a good theoretical physicist today might find it useful to have a wide range of physical viewpoints and mathematical expressions of the same theory available to him.

Richard Feynman, *Nobel Lecture*, 1965

*Formulations of General Relativity.* Facing this title the prospective reader should be thinking, what is there to formulate general relativity (GR)? GR can be formulated in one sentence: GR action functional is the integral of the scalar curvature over the manifold. Everything else that is there to say about GR is the consequence of the Euler–Lagrange equations one obtains by extremising this action, together with the action for matter fields. How can there be a book about ‘formulations’? And why plural? Is there not just the usual Einstein–Hilbert formulation as stated previously?

A more sophisticated reader will know that there are several equivalent formulations of general relativity. There is the usual metric formulation, and then there is an equivalent formulation in terms of tetrads. But this is all well known. GR is about physical consequences of the dynamical postulate that fixes the theory. There may be several equivalent ways to define the dynamics. But this

does not change the physics. So, one formulation is sufficient to unravel all the physics predicted by the theory. The usual metric formulation is by far the most studied and best understood. Why bother about developing any other equivalent formulation? And then why write a book about such unnecessary alternatives?

This is when the two quotes included previously from the Richard Feynman Nobel lecture become relevant. The first is about an empirical observation that theories that are relevant for describing the world around us tend to admit many different equivalent, but not obviously so, reformulations. The example Feynman has in mind is classical electrodynamics, not gravity. Feynman also notices that there is a deep link between the ‘simplicity’ of a theory, and the availability of many different, not manifestly equivalent, descriptions. He goes further and proposes this as the criterion of simplicity. This suggests that one can never fully appreciate the simplicity and beauty of GR without absorbing all the different available formulations of this theory.

The second quote is a different, but not unrelated, thought. There may be equivalent formulations of a theory, all leading to the same physical predictions. But such reformulations may be inequivalent if one decides to generalise. The example of most relevance for Feynman is the Hamiltonian and Lagrangian description of classical mechanics. The quantum generalisation of the Hamiltonian description leads to the usual operator formalism for quantum theory. The generalisation of the Lagrangian description leads to path integrals, which is arguably one of Feynman’s main contributions to physics. These two equivalent formulations of classical mechanics are certainly not equivalent in terms of the new structures that can be generated from them. The same may well apply to gravity. We do not yet know which of the many available formulations of gravity will lead to the next big step in the quest for understanding the world around us.

So, the purpose of this book is to describe all the ‘equivalent’ formulations of general relativity that are known to the author, and that also put the geometry of differential forms and fibre bundles at the forefront of the description of gravity. What is meant by a ‘formulation’ here is a Lagrangian description, in which the dynamical equations are obtained by extremising the corresponding action. This gives us the most economic way of defining the theory.

Some of these equivalent formulations will likely be known to many readers. In particular, this is the already mentioned formulation in terms of tetrads. If this was the complete list, there would be no good reason to write this book.

What is much less known, and what really motivated this author to embark on the present project, is that there are some special features of GR in four spacetime dimensions. These special features are related to coincidences that occur precisely in four dimensions. Thus, in any dimension the Riemann curvature can be viewed as a matrix mapping antisymmetric rank-two tensors again into such tensors. And in four dimensions one also has the Hodge star operator that maps antisymmetric rank-two tensors into such tensors. One can ask how these two operations are related or compatible. It is then a simple to check but deep fact that a

metric is Einstein if and only if these two operations commute. This fact leads to a whole series of *chiral* formulations of four-dimensional GR that have no analogs in higher spacetime dimensions. It is the development of these formulations, and contrasting them with the more known ones, that will occupy us for the large part of this book. There is no coherent account of these developments in the literature, certainly not in any book on GR. It is our desire to make such a coherent account available that was one of the main motivations for writing this monograph.

Another motivation for writing this exposition was our desire to promote the formalism(s) for GR that place the differential forms rather than metrics at the forefront. Differential forms are arguably the simplest and most natural geometric objects that can be placed on a smooth manifold, and are certainly simpler objects than a metric. It turns out to be possible to describe GR using the powerful calculus of differential forms and fibre bundles, which is largely due to Élie Cartan (see Chapter 1 for more on this). This book is in particular aimed at giving an exposition of the possible formalisms that achieve this.

A related theme is that of spinors and spinorial description. As is well known, and as we will also emphasise in the book, spinors and differential forms are essentially the same thing, with the Dirac operator being intimately related to the exterior derivative operator. This means that as soon as differential forms are being used as variables to describe the theory, the description has an interesting spinor translation. Viewed in this way, the kinetic operators arising in the field equations of formulations that use differential forms are various versions of the Dirac operator. This becomes especially pronounced in the so-called first-order formulations where field equations are first order in derivatives. This spinor aspect of gravity (and, as we shall see, Yang–Mills theory too), absent in the usual metric description, is another unifying theme of this book. In addition, the spinor description of gravity simplifies link to some recent developments in the field of scattering amplitudes, as we will touch on.

The more familiar of formalisms that use differential forms rather than metrics is that of tetrad (or vielbein, or moving frame or soldering form) introduced by Cartan. Historically, this formalism was first discovered in the context of two dimensions by the French mathematician Jean-Gaston Darboux (Cartan's PhD supervisor) in the late nineteenth century. It is particularly powerful in this context, as the two 1-forms that encode the metric information can be combined into a single complex-valued one-form on the manifold. This is related to the fact that any 2-manifold is a complex manifold. There is no direct analog of such a complexification trick in four dimensions because there is no longer a unique choice of an almost complex structure. But one gets a computationally powerful formalism in four dimensions via chiral formulations referred to above. These formulations, in the case of Lorentzian signature, bring into play complex-valued objects and in a certain sense provide the analog of the complexification trick that works so well in two dimensions. They also make a link to the twistor description of gravity, as we shall learn.

Our final introductory remark is about Einstein's cornerstone idea that gravity is geometry. At the time when Einstein formulated his theory, the only geometry available to him was the Riemannian geometry of metrics, described via the tensor calculus of Ricci and Levi-Civita. Einstein learned this mathematics guided by his friend and classmate Marcel Grossmann. It is thus no surprise that GR was formulated in the language of Riemannian geometry and tensor calculus. It is still being developed and also taught to graduate students in that way. However, already at the time of Einstein's formulation of GR, Élie Cartan was developing a very different type of geometry, the geometry in which the key role was played by differential forms and connections. His works, and works of those around him, strongly influenced the subject of differential geometry, and it is now far more rich and sophisticated than it was 100 years ago. The Riemannian geometry is now only its relatively small corner. This discussion is related to the theme of the present book because various different formulations of GR that we develop place various different geometric constructions at the forefront. In particular, the geometry of fibre bundles plays a much more important role than it does in the usual description of GR. It is thus certainly true that gravity continues to be geometry in the developments on this book, it is only that the word geometry is being understood more broadly than in the metric GR context. We do not yet know which of these 'geometries' is more fundamental than others, but a good researcher will certainly want to keep his/her mind open and learn all the available options.

The target audience for this book are postgraduate students interested in gravity, as well as already established researchers. To give encouraging words to the first audience, the author would like to recall his own experience as a student. This author remembers very distinctly that it was easiest to study, understand and prepare for exams on classical mechanics by reading Vladimir Arnold's book on the subject. And Paul Dirac's book played a similar role for quantum mechanics. Both books present their respective subjects in a beautiful and logical way, and both are inspired by mathematics. The moral here is that there are some students that learn best by understanding the overall logic of the formalism first, and only then embark on applications and problem solving. This is certainly not a universal way to learn, and most likely not the way to approach the subject for the first time. But it was important to the present author in his time as a student to have accounts of the usual subjects that concentrate more on the overall logic and the mathematical formalism, rather than on concrete problems that can be solved. The author hopes that there are similar minds out there, and that the present exposition will help such students to understand what GR is about.

In terms of the specialised knowledge that is required to understand this book, we do not assume any more than is usually assumed for graduate-level courses. Familiarity with concepts of differential geometry is desirable, but the aspects of this subject that are required to understand the present text are reviewed



in the first chapter. So, a good graduate student should be able to follow this exposition without too much difficulty.

Thus, this book is mainly about different possible formalisms for doing calculations with GR, rather than about different possible physical consequences of this theory. So, this book certainly does not compete with the standard textbook expositions of GR, and the student must also study these more standard sources to understand the physics as predicted by general relativity. Excellent books on this subject that became the standard sources are *General Relativity* by R. Wald and *Spacetime and Geometry: An Introduction to General Relativity* by S. Carroll for GR in general and *Physical Foundations of Cosmology* by V. Mukhanov for applications to cosmology.

For the experienced researchers, the author suggests this book as a source on aspects of GR that are important about this theory, especially in four spacetime dimensions, but are not covered in any standard book on the subject. Thus, the book can be used as a compendium on different available formalisms for GR, as well as on some less standard aspects of geometry that are required to develop these formulations. Additional motivations for why different formulations of GR may lead to new developments and/or new generalisations are given in the concluding chapter.

We end by explaining why it is the quote from Gregory Skovoroda that we chose to be an epigraph for this whole exposition. First, the author is a Ukrainian, and it gives him a distinct pleasure to be able to quote Skovoroda, who was a deep thinker years ahead of his times, and who is still relevant today. He is almost unknown in the West, and maybe one of the readers will remember the name, and read his texts.

Second, we aim here to explain only simple, but in our view important things about GR in four spacetime dimensions. There is much more that can be said, and there is a great wealth of physical phenomena that the theory predicts and describes, and that we omit. Not because they are unimportant – on the contrary, they are the reason why physicists learn the subject. But rather because they are unnecessary to understand the overall logic of the theory. It is this overall logic and the facts likely needed to ‘move to the unknown’ that will concern us in the present book. And so we focus here only on things necessary to understand the overall logic of gravity, and hence only on things simple. We hope the reader will take this as a word of encouragement to follow the development of different formalisms described here.

Finally, I would like to thank my collaborators, from whom I learned a lot and without whose insights this book would not exist. Thanks in particular to Joel Fine, Yannick Herfray, Carlos Scarinci and Yuri Shtanov. Thanks also go to my family for their support to ‘papa’ working on his ‘kniga’.

Cambridge University Press  
978-1-108-48164-9 — Formulations of General Relativity  
Kirill Krasnov  
Frontmatter  
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