Introduction to the Second Edition

We had two main aims in writing this edition:

- Be more self-contained where possible. For instance, we have added brief overviews of Coxeter groups and root systems, and given some more details about the theory of algebraic groups.
- While retaining the same level of exposition as in the first edition, we have given a more complete account of the representation theory of finite groups of Lie type.

In view of the second aim, we have added the following topics to our exposition:

- We cover Ree and Suzuki groups extending our exposition of Frobenius morphisms to the more general case of Frobenius roots.
- We have added to Harish-Chandra theory the topic of Hecke algebras and given as many results as we could easily do for fields of arbitrary characteristic prime to q, in view of applications to modular representations.
- We have added a chapter on the computation of Green functions, with a brief review of invariant theory of reflection groups, and a chapter on the decomposition of unipotent Deligne–Lusztig characters.

Acknowledgements

In addition to the people we thank in the introduction to the first edition, we have benefitted from the input of younger colleagues such as Cédric Bonnafé, Olivier Dudas, Meinolf Geck, Daniel Juteau, and Raphaël Rouquier, and we give special thanks to Gunter Malle for a thorough proofreading of this edition.

We also thank the many people who pointed out to us misprints and other errors in the first edition.

From the Introduction to the First Edition

These notes follow a course given at the Paris VII university during the spring semester of academic year 1987–88. Their purpose is to expound basic results in the representation theory of finite groups of Lie type (a precise definition of this concept will be given in the chapter "Rationality, Frobenius").

Let us start with some notations. We denote by $\mathbb{F}_q$ a finite field of characteristic p with q elements (q is a power of p). The typical groups we will look at are the linear, unitary, symplectic, orthogonal, ..., groups over $\mathbb{F}_q$. We will consider these groups as the subgroups of points with coefficients in $\mathbb{F}_q$ of the corresponding groups over an algebraic closure $\overline{\mathbb{F}}_q$ (which are algebraic reductive groups). More precisely, the group over $\mathbb{F}_q$ is the set of fixed points of the group over $\overline{\mathbb{F}}_q$ under an endomorphism F called the Frobenius endomorphism; this will be explained in the chapter "Rationality, Frobenius". In the following paragraphs of this introduction we will try to describe, by some examples, a sample of the methods used to study the complex representations of these groups.

Induction from Subgroups

Let us start with the example where $G = \mathbf{GL}_n(\mathbb{F}_q)$ is the general linear group over $\mathbb{F}_q$. Let T be the subgroup of diagonal matrices; it is a subgroup of the group B of upper triangular matrices, and there is a semidirect product decomposition $B = U \rtimes T$, where U is the subgroup of the upper triangular matrices which have all their diagonal entries equal to 1. The representation theory of T is easy, since it is a commutative group (actually isomorphic to a product of n copies of the multiplicative group $\mathbb{F}_q^\times$). Composition with the natural homomorphism from B to T (quotient by U) lifts representations of T to representations of B. Inducing these representations from B to the whole of the general linear group gives representations of G (whose irreducible constituents are called "principal series representations"). More generally, we can replace T with a group L of block-diagonal matrices, B with the group of corresponding upper block-triangular matrices P, and we have a semi-direct product decomposition (called a Levi decomposition) $P = V \rtimes L$, where V is the subgroup of P whose diagonal blocks are identity matrices; we may as before induce from P to G representations of L lifted to P. The point of this method is that L is isomorphic to a direct product of general linear groups of smaller degrees than n. We thus have an inductive process to get representations of G if we know how to decompose induced representations from P to G. This approach has been developed in the works of Harish-Chandra, Howlett and Lehrer, and is introduced in the chapter "Harish-Chandra Theory".

Cohomological Methods

Let us now consider the example of $G = U_n$, the unitary group. It can be defined as the subgroup of matrices $A \in \mathbf{GL}_n(\mathbb{F}_{q^2})$ such that ${}^tA^{[q]} = A^{-1}$, where $A^{[q]}$ denotes the matrix whose entries are those of A raised to the qth power. It is thus the subgroup of $\mathbf{GL}_n(\overline{\mathbb{F}}_q)$ consisting of the fixed points of the endomorphism $F : A \mapsto ({}^tA^{[q]})^{-1}$.

A subgroup L of block-diagonal matrices in U_n is again a product of unitary groups of smaller degree. But this time we cannot construct a bigger group P having L as a quotient. More precisely, the group $\mathbf{V}$ of upper block-triangular matrices with entries in $\overline{\mathbb{F}}_q$ and whose diagonal blocks are the identity matrix has no fixed points other than the identity under F.

To get a suitable theory, Harish-Chandra's construction must be generalised; instead of inducing from $V \rtimes L$ to G, we construct a variety attached to $\mathbf{V}$ on which both L and G act with commuting actions, and the cohomology of that variety with ℓ-adic coefficients gives a (virtual) bi-module which defines a "generalised induction" from L to G. This approach, due to Deligne and Lusztig, will be developed in the chapters from "ℓ-adic Cohomology" to "Geometric Conjugacy and Lusztig Series".

Gelfand–Graev Representations

Using the above methods, a lot of information can be obtained about the characters of the groups $\mathbf{G}(\mathbb{F}_q)$, when $\mathbf{G}$ has a connected centre. The situation is not so clear when the centre of $\mathbf{G}$ is not connected. In this case one can use the Gelfand–Graev representations, which are obtained by inducing a linear character "in general position" of a maximal unipotent subgroup (in $\mathbf{GL}_n$ the subgroup of upper triangular matrices with ones on the diagonal is such a subgroup). These representations are closely tied to the theory of regular unipotent elements. They are multiplicity-free and contain rather large cross-sections of the set of irreducible characters, and so give useful additional information in the nonconnected centre case. (In the connected centre case, they are linear combinations of Deligne–Lusztig characters.)

For instance, in $\mathbf{SL}_2(\mathbb{F}_q)$ they are obtained by inducing a nontrivial linear character of the group of matrices of the form $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$: such a character corresponds to a nontrivial additive character of $\mathbb{F}_q$; there are two classes of such characters under $\mathbf{SL}_2(\mathbb{F}_q)$, which corresponds to the fact that the centre of $\mathbf{SL}_2$ has two connected components (its two elements).

The theory of regular elements and Gelfand–Graev representations is expounded in chapter "Regular Elements; Gelfand–Graev Representations", with, as an application, the computation of the values of all irreducible characters on regular unipotent elements.

Acknowledgements

We would like to thank the "équipe des groupes finis" and the mathematics department of the École Normale Supérieure, who provided us with a stimulating working environment, and adequate facilities for composing this book. We thank also the Paris VII university, which gave us the opportunity to give the course which started this book. We thank all those who carefully read the earlier drafts and suggested improvements, particularly Michel Enguehard, Guy Rousseau, Jean-Yves Hée and the editor. We thank Michel Broué and Jacques Tits, who provided us with various ideas and information, and above all George Lusztig, who invented most of the theory.