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Excerpt
More Information

1 Classical Relativistic Point Particles

In this chapter, we review classical relativistic point particles and set out our conventions and notation. Readers familiar with this material can skip through the chapter and use it as a reference when needed.

1.1 Minkowski Space

According to Einstein's theory of special relativity, space and time are combined into 'space-time', which is modelled by Minkowski space $\mathbb{M}$.[1] The elements $P, Q, \ldots \in \mathbb{M}$ are called *events*. We leave the dimension D of space-time unspecified. Minkowski space is homogeneous and thus has no preferred origin, which makes it a point space (affine space) rather than a vector space (linear space). However, displacements relating events P, Q are vectors,

$$x = \overrightarrow{PQ} \in \mathbb{R}^D, \tag{1.1}$$

and once we choose a point $O \in \mathbb{M}$ as the origin of our coordinate system there is a one-to-one correspondence between events P and position vectors

$$x_P = \overrightarrow{OP}. \tag{1.2}$$

The components

$$(x^\mu)_{\mu=0,1,\ldots,D-1} = (x^0, \vec{x}) \ , \quad \vec{x} = (x^i)_{i=1,\ldots,D-1} \tag{1.3}$$

of vectors $x \in \mathbb{R}^D$ provide linear coordinates on $\mathbb{M}$. We assume that $x^i = 0$ is the world-line of an inertial (force-free) observer, so that $x^0 = ct$ is proportional to the time t measured in the associated inertial system, while x^i provide linear coordinates on space. We will normally use natural units where we set the speed of light to unity, $c = 1$.[2]

To measure the distance between events, we use the indefinite scalar product

$$x \cdot y = \eta_{\mu\nu} x^\mu y^\nu, \tag{1.4}$$

on the vector space $\mathbb{R}^D$, with Gram matrix

$$\eta = (\eta_{\mu\nu}) = \begin{pmatrix} -1 & \vec{0}^{\,T} \\ \vec{0} & \mathbb{1}_{D-1} \end{pmatrix}. \tag{1.5}$$

[1] For brevity's sake we will use 'Minkowksi space' instead of 'Minkowski space-time'.
[2] Conventions and units are reviewed in Appendices A and B, respectively.

Note that we use the *mostly plus convention* for the metric η. Since the metric allows us to identify the vector space $\mathbb{R}^D$ with its dual, vectors have covariant components x_μ as well as contravariant components x^μ, which are related by *raising and lowering indices* $x_\mu = \eta_{\mu\nu} x^\nu$, and $x^\mu = \eta^{\mu\nu} x_\nu$, where Einstein's summation convention is understood. The corresponding line element on Minkowski space is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\vec{x}^2. \tag{1.6}$$

The most general class of transformations which preserve this line element (its *isometries*) are the Poincaré transformations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{1.7}$$

where $a = (a^\mu) \in \mathbb{R}^D$ and where $\Lambda = (\Lambda^\mu{}_\nu)$ is an invertible $D \times D$ matrix satisfying

$$\Lambda^T \eta \Lambda = \eta. \tag{1.8}$$

The matrices Λ describe *Lorentz transformations*, which are the most general linear transformations preserving the metric. The Lorentz transformations form a Lie group of dimension $\frac{1}{2}D(D-1)$, called the *Lorentz group* $\mathrm{O}(1, D-1)$. Elements $\Lambda \in \mathrm{O}(1, D-1)$ have determinant $\det \Lambda = \pm 1$, and satisfy $|\Lambda^0{}_0| \geq 1$. The matrices with $\det \Lambda = 1$ form a subgroup $\mathrm{SO}(1, D-1)$. This subgroup still has two connected components, since $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. The connected component containing the unit matrix $\mathbb{1} \in \mathrm{O}(1, D-1)$ is the *connected* or *proper orthochronous Lorentz group* $\mathrm{SO}_0(1, D-1)$. The corresponding Lie algebra is $\mathfrak{so}(1, D-1)$.

The Lorentz group and translation group combine into the *Poincaré group*, or inhomogeneous Lorentz group, $\mathrm{IO}(1, D-1)$, which is a Lie group of dimension $\frac{1}{2}D(D+1)$. Since Lorentz transformations and translations do not commute, the Poincaré group is not a direct product. The composition law

$$(\Lambda, a) \circ (\Lambda', a') = (\Lambda\Lambda', a + \Lambda a') \tag{1.9}$$

shows that the Lorentz group operates on the translation group by the fundamental or vector representation. Therefore, the Poincaré group is the *semi-direct product* of the Lorentz and translation group,

$$\mathrm{IO}(1, D-1) = \mathrm{O}(1, D-1) \ltimes \mathbb{R}^D. \tag{1.10}$$

Since the Minkowski metric (1.6) is defined by an indefinite scalar product, the square-distance or square-norm $x^2 = x \cdot x$ can be positive, zero, or negative. For terminological simplicity, we will henceforth refer to $x^2 = x \cdot x$ as the *norm* or *length* of x, omitting the qualifier 'square'. This convention will be applied whenever we deal with an indefinite scalar product.

Vectors are classified as *time-like*, *light-like* (also called *null*), or *space-like* according to their norm:

$$\begin{aligned} x \text{ time-like} &\Leftrightarrow x \cdot x < 0, \\ x \text{ light-like} &\Leftrightarrow x \cdot x = 0, \\ x \text{ space-like} &\Leftrightarrow x \cdot x > 0. \end{aligned}$$

Since signals can only travel with speed $v \leq 1 (= c)$, this encodes information about the causal relations between events. Two events P, Q are called time-like, light-like, or space-like relative to each other, if the displacement vector $x = \overrightarrow{PQ}$ is time-like, light-like, or space-like, respectively. Only non-space-like events can be causally related, and their causal order is invariant under orthochronous Poincaré transformations, which exclude the time inversion $T : t \to -t, \vec{x} \to \vec{x}$. Since some particle interactions are not invariant under the space inversion $P : t \to t, \vec{x} \to -\vec{x}$, the symmetry group relevant for particle physics is the proper orthochronous Poincaré group $\mathrm{SO}_0(1, D-1) \ltimes \mathbb{R}^D$. This is the connected component of the unit element of the full Poincaré group, which has three further connected components which contain T, P, and their product TP.

1.2 Particles

The fundamental constituents of matter are usually modelled as particles, that is, as objects that are localised and can be characterised by a small number of parameters, such as mass, spin, and charges. While some particles are bound states of others, the standard model of particle physics is based on a list of particles, assumed to be elementary in the sense that they do not have constituents and, therefore, no internal excitations. In classical mechanics, such particles are modelled as mathematical points. The motion of such a point particle, or *particle* for short, is described by a parametrised curve called the *world-line*. If we restrict ourselves to inertial frames, it is natural to choose the coordinate time t as the curve parameter. Then, the world-line of a particle is a parametrised curve

$$C : I \to \mathbb{M} \; : t \mapsto x(t) = (x^\mu(t)) = (t, \vec{x}(t)), \tag{1.11}$$

where $I \subset \mathbb{R}$ is the time interval for which the particle is observed. $I = \mathbb{R}$ is included as a limiting case.

The *velocity* of a particle relative to an inertial frame is

$$\vec{v} = \frac{d\vec{x}}{dt}, \tag{1.12}$$

and $v = \sqrt{\vec{v} \cdot \vec{v}} \geq 0$ is the speed. Since t and $\vec{v}$ are not covariant quantities (Lorentz tensors), it is useful to formulate relativistic mechanics using the Lorentz vector x^μ and its derivatives with respect to a curve parameter which is a Lorentz scalar. This works differently for massive and for massless particles.

The inertial *mass* m of a particle measures its resistance against a change of velocity. Massive particles, $m > 0$, propagate with velocities $v < 1$ and have time-like world-lines, that is world-lines where the tangent vector is time-like everywhere. Massless particles, $m = 0$, propagate with velocity $v = 1$ and have light-like world-lines. Poincaré symmetry also admits *tachyons*, that is, particles with negative mass-squared, $m^2 < 0$, which propagate with velocity $v > 1$ and have space-like world-lines. Such tachyons are discarded because they would allow a-causal effects, such as sending signals backwards in time. In quantum field theory, tachyons are re-interpreted as indicating instabilities resulting from expanding a theory around a local maximum of the potential. This is a physical effect and does not involve particles propagating with superluminal speed (see Section 7.7).

For massive particles, we can use the *proper time* τ as a curve parameter. Infinitesimally, the relation between proper time and coordinate time is

$$-d\tau^2 = [-dt^2 + d\vec{x}^2]_{|C} = \left[-1 + \left(\frac{d\vec{x}}{dt}\right)^2\right] dt^2 \Rightarrow d\tau = dt\sqrt{1 - \vec{v}^2}. \tag{1.13}$$

Since τ is, by construction, Lorentz invariant, the *relativistic velocity*

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{dt}\frac{dt}{d\tau}\right) = \frac{1}{\sqrt{1 - \vec{v}^2}}(1, \vec{v}) \tag{1.14}$$

is a Lorentz vector. Moreover, it is a time-like *unit tangent vector* to the world-line, since $\dot{x}^2 = \dot{x}^\mu \dot{x}_\mu = -1$. The norm is in particular constant, which makes τ an *affine curve parameter*. The name 'affine' curve parameter reflects the fact that such curve parameters are unique up to affine transformations, $\tau \mapsto a\tau + b$, $a, b \in \mathbb{R}$, $a \neq 0$.

By further differentiation, we obtain the *relativistic acceleration*,

$$a^\mu = \ddot{x}^\mu. \tag{1.15}$$

Newton's first law states that force-free particles are unaccelerated relative to inertial frames.

The *relativistic momentum* of a particle is

$$p^\mu = m\dot{x}^\mu = (p^0, \vec{p}) = \left(\frac{m}{\sqrt{1 - \vec{v}^2}}, \frac{m\vec{v}}{\sqrt{1 - \vec{v}^2}}\right). \tag{1.16}$$

The component $p^0 = E$ is the total energy of the particle. The norm of p^μ is minus its mass squared

$$p^\mu p_\mu = -m^2 = -E^2 + \vec{p}^2. \tag{1.17}$$

Note the minus sign which is due to us using the mostly plus convention for the metric.

Force-free particles propagate with constant velocity, which means that their world-lines are straight lines. The relativistic version of *Newton's second law* states that motion under a force is determined by the equation

$$\frac{dp^\mu}{d\tau} = m\frac{d^2x^\mu}{d\tau^2} = f^\mu, \tag{1.18}$$

where the Lorentz vector f^μ is the relativistic force. Note that we assume that the mass m is constant, which is satisfied for stable elementary particles but may not hold in other applications of relativistic mechanics (e.g., for the motion of a rocket which expels fuel).

1.3 A Non-covariant Action Principle for Relativistic Particles

The equations of motion of all fundamental physical theories can be obtained from variational principles. In this approach, a theory is defined by specifying its *action* which is a functional on the configuration space. The equations of motion are the Euler–Lagrange equations obtained by imposing that the action is invariant under infinitesimal variations of the path, with the initial and final position kept fixed.

For a point particle, the configuration space is parametrised by its position $\vec{x}$ and velocity $\vec{v}$. The action functional takes the form

$$S[\vec{x}] = \int dt\, L(\vec{x}(t), \vec{v}(t)). \tag{1.19}$$

In principle, the Lagrangian L can have an explicit dependence on time, corresponding to a time-dependent potential or external field. In fundamental theories, we assume the invariance of the field equations under time-translations, which forbids an explicit time dependence of L.

The action for a free, massive, relativistic particle is proportional to the proper time along the world-line, and given by minus the product of its mass and the proper time:

$$S = -m \int dt \sqrt{1 - \vec{v}^2}. \tag{1.20}$$

The minus sign has been introduced so that L has the conventional form $L = T - V$ where T is the part quadratic in time derivatives, that is, the *kinetic energy*. The remaining part V is the *potential energy*. We work in units where the speed of light and the reduced Planck constant have been set to unity, $c = 1, \hbar = 1$. In such *natural units* the action S is dimensionless. To verify that the action principle reproduces the equation of motion (1.18), we consider the motion $\vec{x}(t)$ of a particle between the initial postion $\vec{x}_1 = \vec{x}(t_1)$ and the final position $\vec{x}_2 = \vec{x}(t_2)$. Then, we compute the first order variation of the action under infinitesimal variations $\vec{x} \to \vec{x} + \delta\vec{x}$, which are arbitrary, except for the boundary conditions $\delta\vec{x}(t_i) = 0, i = 1, 2$ (see Figure 1.1). To compare the initial and deformed path we Taylor expand in $\delta\vec{x}(t)$:

$$S[\vec{x}(t) + \delta\vec{x}(t)] = S[\vec{x}(t), \vec{v}(t)] + \delta S[\vec{x}(t), \vec{v}(t)] + \cdots \tag{1.21}$$

where the omitted terms are of quadratic and higher order in $\delta\vec{x}(t)$. The equations of motion are found by imposing that the first variation δS vanishes.

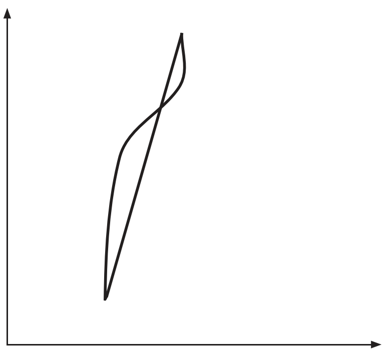

Fig. 1.1 The action principle selects the paths for which the action is stationary under variation. The endpoints are kept fixed.

Practical manipulations are most easily performed using the following observations:

1. The variation δ *acts like a derivative*. For example, for a function $f(\vec{x})$ we have the chain rule
$$\delta f = \partial_i f \delta x^i, \tag{1.22}$$
as is easily verified by Taylor expanding $f(\vec{x} + \delta\vec{x})$. Similarly, sum, constant factor, product, and quotient apply, for example, $\delta(fg) = \delta f g + f\delta g$.
2. $\vec{v} = \frac{d\vec{x}}{dt}$ is not an independent quantity, and therefore
$$\delta\vec{v} = \delta\frac{d\vec{x}}{dt} = \frac{d}{dt}\delta\vec{x}. \tag{1.23}$$
3. To find δS, we need to collect all terms proportional to $\delta\vec{x}$. Derivatives acting on $\delta\vec{x}$ have to be removed through integration by parts which creates boundary terms. If such boundary terms are not automatically zero, we must impose that they vanish which restricts the class of configurations which qualify as solutions.

Solving the variational problem for the action (1.20), we obtain the equation of motion
$$\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} = \frac{d}{dt}\vec{p} = \vec{0}, \tag{1.24}$$
which is equivalent to (1.18) in the absence of forces, $f^\mu = 0$.

Remark: When performing the variation without specifying the Lagrangian L, one obtains the Euler–Lagrange equations
$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i} = 0. \tag{1.25}$$
For $L = -m\sqrt{1-\vec{v}^2}$ this is easily seen to give (1.24).[3]

Exercise 1.3.1 Verify that the variation of (1.20) takes the form
$$\delta S = -\int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{mv_i}{\sqrt{1-\vec{v}^2}}\right)\delta x^i dt + \left.\frac{mv_i}{\sqrt{1-\vec{v}^2}}\delta x^i\right|_{t_1}^{t_2}. \tag{1.26}$$
Does the boundary term impose any conditions on the dynamics?

Exercise 1.3.2 Verify that (1.24) is equivalent to (1.18) with $f^\mu = 0$. Then, extend this to the case where a force term is present. This requires one to know, or to derive, the relation between the relativistic force f^μ and the non-relativistic expression $\vec{F}$. Show that the non-covariant version of Newton's second law,
$$\frac{d\vec{p}}{dt} = \vec{F}, \quad \text{where} \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2}} \tag{1.27}$$
is equivalent to (1.18).

[3] In my opinion, it is more natural, convenient, and insightful to obtain the equations of motion for a given concrete theory by varying the corresponding action, as done above, instead of plugging the Lagrangian into the Euler–Lagrange equations. This procedure reminds one that there may be boundary terms that one has to worry about, as we will see when replacing particles by strings.

1.4 Canonical Momenta and Hamiltonian

We now turn to the Hamiltonian description of the relativistic particle. In the Lagrangian formalism we use the configuration space variables $(\vec{x}, \vec{v}) = (x^i, v^i)$. In the Hamiltonian formalism, the velocity $\vec{v}$ is replaced by the canonical momentum

$$\pi^i := \frac{\partial L}{\partial v_i}. \tag{1.28}$$

For the Lagrangian $L = -m\sqrt{1-\vec{v}^2}$, the canonical momentum agrees with the kinetic momentum, $\vec{\pi} = \vec{p} = (1-\vec{v}^2)^{-1/2} m\vec{v}$. However, conceptually canoncial and kinetic momentum are different quantities. A standard example where the two quantities are not equal is a charged particle in a magnetic field (see Section 13.6, i.p. formula (13.169)).

The Hamiltonian $H(\vec{x}, \vec{\pi})$ is obtained from the Lagrangian $L(\vec{x}, \vec{v})$ by a Legendre transformation:

$$H(\vec{x}, \vec{\pi}) = \vec{\pi} \cdot \vec{v} - L(\vec{x}, \vec{v}(\vec{x}, \vec{\pi})). \tag{1.29}$$

For $L = -m\sqrt{1-\vec{v}^2}$ the Hamiltonian is equal to the total energy:

$$H = \vec{\pi} \cdot \vec{v} - L = \vec{p} \cdot \vec{v} - L = \frac{m}{\sqrt{1-\vec{v}^2}} = p^0 = E. \tag{1.30}$$

Describing relativistic particles using the action (1.20) has the following disadvantages:

- We can describe massive particles, but photons, gluons, and the hypothetical gravitons underlying gravity are believed to be massless. How can we describe massless particles?
- The independent variables $\vec{x}, \vec{v}$ are not Lorentz vectors. Therefore, our formalism lacks manifest Lorentz covariance. How can we formulate an action principle that is Lorentz covariant?
- We have picked a particular curve parameter for the world-line, namely the inertial time with respect to a Lorentz frame. While this is a natural choice, 'physics', that is, observational data, cannot depend on how we label points on the world-line. How can we formulate an action principle that is manifestly covariant with respect to reparametrisations of the world-line?

We will answer these questions in reverse order.

1.5 Length, Proper Time, and Reparametrisations

To prepare for the following, we first discuss general curve parameters and reparametrisations. Consider a smooth parametrised curve in Minkowski space,

$$C : I \ni \sigma \longrightarrow x^\mu(\sigma) \in \mathbb{M}, \tag{1.31}$$

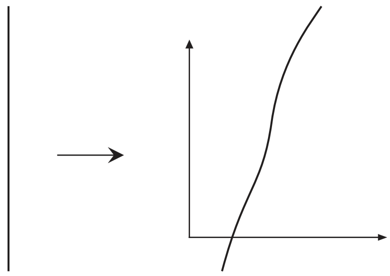

Fig. 1.2 The world-line of a particle is described by a parametrised curve, that is, by a map from a parameter interval into space-time. Physical quantities do not depend on the parametrisation.

where σ is an arbitrary curve parameter, taking values in an interval $I \subset \mathbb{R}$ (Figure 1.2).

We can *reparametrise* the curve by introducing a new curve parameter $\tilde{\sigma} \in \tilde{I}$ which is related to σ by an invertible map

$$\sigma \to \tilde{\sigma}(\sigma) , \quad \text{where} \quad \frac{d\tilde{\sigma}}{d\sigma} \neq 0. \tag{1.32}$$

While $C : I \to \mathbb{M}$ and $\tilde{C} : \tilde{I} \to \mathbb{M}$ are different maps, they have the same image in $\mathbb{M}$ and we regard them as different descriptions (parametrisations) of the same curve. The quantity $d\tilde{\sigma}/d\sigma$ is the *Jacobian* of this reparametrisation.

Often, one imposes the stronger condition

$$\frac{d\tilde{\sigma}}{d\sigma} > 0, \tag{1.33}$$

which means that the orientation (direction) of the curve is preserved.

The tangent vector field of the curve is

$$x'^{\mu} := \frac{dx^{\mu}}{d\sigma}. \tag{1.34}$$

A curve $C \ : I \to \mathbb{M}$ is called space-like, light-like, or space-like if its tangent vector field is space-like, light-like, or space-like, respectively, for all $\sigma \in I$. This property is reparametrisation invariant.

For a space-like curve, $I = [\sigma_1, \sigma_2] \to \mathbb{M}$, the *length* (or 'proper length') is defined as

$$L = \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{\eta_{\mu\nu} \frac{dx^{\mu}}{d\sigma} \frac{dx^{\nu}}{d\sigma}}. \tag{1.35}$$

For a time-like curve, we can define a 'length' by

$$\tau(\sigma_1, \sigma_2) = \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^{\mu}}{d\sigma} \frac{dx^{\nu}}{d\sigma}}, \tag{1.36}$$

and this quantity is precisely the proper time for a particle that has this curve as its world-line. We note that the proper length and proper time are distinguished affine curve parameters, characterised by the tangent vector field having unit norm. For light-like curves there is no analogous quantity, but we will see that there still is a distinguished class of affine curve parameters for the world-lines of massless particles.

Exercise 1.5.1 Verify that the length (1.35) of a space-like curve is reparametrisation invariant. Why does this not depend on whether the reparametrisation preserves the orientation of the curve?

Exercise 1.5.2 Show that the tangent vector field $\frac{dx^\mu}{d\tau}$ for the curve parameter τ defined by (1.36) has norm $\dot{x}^2 = -1$, thus verifying that τ is the proper time.

1.6 A Covariant Action for Massive Relativistic Particles

Using the concepts of the previous section, we introduce the following action:

$$S[x] = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}. \tag{1.37}$$

Up to the constant factor $-m$, the action is the proper time for the motion of the particle along the world-line. We use an arbitrary curve parameter σ, and configuration space variables $(x, x') = (x^\mu, x'^\mu)$, which transform covariantly under Lorentz transformations. The action (1.37) has the following symmetries (invariances):

- The action is invariant under reparametrisations $\sigma \to \tilde{\sigma}(\sigma)$ of the world-line.
- The action is invariant under Poincaré transformations of space-time.

To verify that the new action (1.37) leads to the same field equations as (1.20), we perform the variation $x^\mu \to x^\mu + \delta x^\mu$ and obtain:

$$\frac{\delta S}{\delta x^\mu} = 0 \Leftrightarrow \frac{d}{d\sigma}\left(\frac{m\, x'^{\,\mu}}{\sqrt{-x' \cdot x'}}\right) = 0. \tag{1.38}$$

To get the physical interpretation, we choose the curve parameter σ to be the proper time τ:

$$\frac{d}{d\tau}\left(m \frac{dx^\mu}{d\tau}\right) = m\ddot{x}^\mu = 0, \tag{1.39}$$

where a ‘dot’ denotes the derivative with respect to proper time. This is indeed (1.18) with $f^\mu = 0$.

The general solution of this equation, which describes the motion of a free massive particle in Minkowski space is the straight world-line

$$x^\mu(\tau) = x^\mu(0) + \dot{x}^\mu(0)\tau. \tag{1.40}$$

Remark: *Reparametrisations vs Diffeomorphisms.* Reparametrisation invariance is also referred to as diffeomorphism invariance. We use the term reparametrisation, rather than diffeomorphism, to emphasise that we interpret the map $\sigma \mapsto \tilde{\sigma}$ *passively*, that is, as a change parametrisation. In contrast, an *active* transformation maps a given point to another point. The expressions for passive and active transformation agree up to an overall minus sign, as we will see in later examples, in particular, in Exercise 5.2.2.

Remark: *'Local' vs 'global' in mathematical and physical terminology.* In mathematics, 'local' refers to statements which hold on open neighbourhoods around each point, whereas 'global' refers to statements holding for the whole space. In contrast, physicists call symmetries 'global' or 'rigid' if the transformation parameters are independent of space-time, and 'local' if the transformation parameters are functions on space-time. In the case of the point particle action, Poincaré transformations are global symmetries, while reparametrisations are local. I will try to reduce the risk of confusion by saying 'rigid symmetry' rather than 'global symmetry', and when a symmetry is referred to as local, it is meant in the physicist's sense. Also, it is common for physicists to talk about statements which are true locally (in the mathematician's sense) but not necessarily true globally, using 'global terminology'.[4]

1.7 Particle Interactions

So far we have considered free particles. Interactions can be introduced by adding terms which couple the particle to external fields. The most important examples are the following:

- If the force f^μ has a potential, $f_\mu = -\partial_\mu V(x)$, then the equation of motion (1.18) follows from the action

$$S = -m \int \sqrt{-\dot{x}^2} d\tau - \int V(x(\tau)) d\tau. \tag{1.41}$$

- If f^μ is the Lorentz force acting on a particle with charge q, that is $f^\mu = qF^{\mu\nu}\dot{x}_\nu$, then the action is

$$S = -m \int \sqrt{-\dot{x}^2} d\tau + q \int A_\mu dx^\mu. \tag{1.42}$$

In the second term, the vector potential A_μ is integrated along the world-line of the particle

$$\int A_\mu dx^\mu = \int A_\mu(x(\tau)) \frac{dx^\mu}{d\tau} d\tau. \tag{1.43}$$

The resulting equation of motion is

$$m\ddot{x}_\mu = qF_{\mu\nu}\dot{x}^\nu, \tag{1.44}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. Equation (1.44) is the manifestly covariant version of the Lorentz force

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{1.45}$$

[4] An example where the distinction between local and global aspects is relevant will be given later in Section 4.2 when we discuss the actions of the conformal Lie algebra and of the conformal group on the string world-sheet.