Stable Lévy Processes via Lamperti-Type Representations

Stable Lévy processes lie at the intersection of Lévy processes and self-similar Markov processes. Processes in the latter class enjoy a Lamperti-type representation as the space-time path transformation of so-called Markov additive processes (MAPs). This completely new mathematical treatment takes advantage of the fact that the underlying MAP for stable Lévy processes can be explicitly described in one dimension and semi-explicitly described in higher dimensions, and uses this approach to catalogue a large number of explicit results describing the path fluctuations of stable Lévy processes in one and higher dimensions.

Written for graduate students and researchers in the field, this book systematically establishes many classical results as well as presenting many recent results appearing in the last decade, including previously unpublished material. Topics explored include first hitting laws for a variety of sets, path conditionings, law-preserving path transformations, the distribution of extremal points, growth envelopes and winding behaviour.

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This book is dedicated to crossing barriers,
not erecting them

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Notation

Below is some of the more commonly used notation that appears throughout the text, which has been thematically grouped for convenience. Reference page numbers are presented in the right-hand column.

Stable distributions

α	stability index	3, 61
ρ	positivity index	12, 61
$\mathcal{A}$	parameter set (α, ρ) for stable distributions	12
$p(x, \alpha, \rho)$	pdf of stable distribution with parameters (α, ρ)	14
$M(z)$	Mellin transform of $p(x, \alpha, \rho)$	17

Lévy processes

(Y, P)	general Lévy process	27
$(\hat{Y}, \mathrm{P})$, $(Y, \hat{\mathrm{P}})$	dual of the Lévy process (Y, P)	41
$\Pi(\mathrm{d}x)$	Lévy measure	5, 28
$\pi(x)$	Lévy density	84
$N(\mathrm{d}t, \mathrm{d}x)$	Poisson point process of jumps	29
$\mathcal{L}$	infinitesimal generator	35
$\mathcal{F}_t$	natural filtration	34
P_t	semigroup	41
Ψ	characteristic exponent	28,
ψ	Laplace exponent	94
$\overline{Y}_t$, $\underline{Y}_t$	running supremum and running infimum	41
Ψ_q^+, Ψ_q^-	Wiener–Hopf factors	183
H, $\hat{H}$	ascending and descending ladder height process	46

$\varsigma,\ \hat{\varsigma}$	lifetime of the ascending and descending ladder height processes	50, 81
$\kappa,\ \hat{\kappa}$	ascending and descending ladder height Laplace exponents	46
$U^{(q)}[f]$	q-resolvent	42
$u^{(q)}$	density of q-resolvent	43
U	subordinator resolvent	50
τ^B	first passage time of a Lévy process into B	42
$\tau_x^-,\ \tau_x^+$	first passage times below and above x	34, 50
ζ	lifetime of killed process	30
$\mathcal{E}_t(\beta)$	exponential martingale	37
$\mathfrak{U},\ W$	variables characterising asymptotic overshoot of a Lévy process at first passage over a threshold tending to infinity	53, 233
$\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$	parametric classes of hypergeometric Lévy processes	84. 88, 93
$\xi^*,\ \Psi^*$	Lévy process underlying stable process killed on entering $(-\infty, 0)$ and its characteristic exponent	123
$\xi^\uparrow,\ \Psi^\uparrow$	Lévy process underlying stable process conditioned to stay positive and its characteristic exponent	131, 132
$\xi^\downarrow,\ \Psi^\downarrow$	Lévy process underlying stable process conditioned to limit to 0 from above and its characteristic exponent	135, 135
$\vec{\xi}_t,\ \vec{\Psi}$	Lévy process underlying censored stable process and its characteristic exponent	137, 143
$\xi,\ \Psi$	Lévy process underlying the radial part of a stable process and its characteristic exponent	145, 147
η	a constant defined from the hypergeometric Lévy process parameters, equal to $1-\beta+\gamma+\hat{\beta}+\hat{\gamma}$	84
$\hat{\theta}$	Cramér number	94
$I(\delta, Y)$	integrated exponential functional of Y	93
χ	shorthand for $1/\delta$	94
$p(x)$	pdf of $I(\delta, Y)$	104, 105
$\mathtt{M}(s)$	Mellin transform of $I(\delta, Y)$	94

Stable Processes

$(X, \mathbb{P})$	stable process	58
$\mathcal{A}$	parameter set (α, ρ) for stable processes	61

$\mathcal{A}^+$	parametric set (α,ρ) for stable processes with positive jumps	214
$\mathtt{p}_t(x)$	density of the stable process issued from 0	60, 361
$\lvert X\rvert$	radial distance from the origin	144
X_t^*	running maximum of absolute value	216
$\mathbb{P}^\uparrow$	law of the stable process conditioned to stay positive	129
$\mathbb{P}^\downarrow$	law of the stable process conditioned to limit to 0 from above	133
$\mathbb{P}^\circ$	stable process conditioned to approach 0 continuously (for $\alpha < d$) or conditioned to avoid the origin (for $d < \alpha$)	308, 314
U^A, u^A	resolvent up to exiting the interval A and its density	157, 165
$U^A_{\{z\}}$, $u^A_{\{z\}}$	resolvent up to exiting the interval $A\backslash\{z\}$ and its density	161, 320
$\mathtt{R}_t$	process reflected in its infimum	176
γ_a	first passage time over threshold γ_a of reflected process	177
$\bar{\mathtt{R}}_t$	running supremum of reflected process	179
$\mathtt{J}_t$	future infimum of stable process	240
$\mathtt{p}_{\overline{X}}(x)$	pdf of the maximum at time 1	194
$\mathcal{M}(x)$	Mellin transform of the maximum at time 1	192
$\eta(t)$	time change in Riesz–Bogdan–Żak transformation	316, 324
$\mathtt{G}(\infty)$	time of closest radial reach to the origin	386
$\overline{m}$	time of furthest radial reach from the origin before hitting the origin	340
$\tau_a^\odot$	first hitting of the sphere of radius a	355
$\mathtt{D}_a$	last passage time of radial distance below a	419
$\tau_a^\oplus$	first entry into the sphere of radius a	365
$\tau_a^\ominus$	first exit from the sphere of radius a	365
θ_t, $\theta_{[a,b]}$	winding numbers of planar stable processes	413
$\mathtt{U}_t$, $\mathtt{U}_{[a,b]}$	upcrossings of one-dimensional stable processes	422

Markov additive processes

E	state space of modulator	287, 300
(ξ, J)	MAP with discrete Markov modulator	287
(ξ, Θ)	MAP with general Markov modulator (usually a $\mathbb{S}^{d-1}$-valued modulator)	300
$\mathcal{G}_t$	MAP filtration	300

$\mathbf{P}_{x,i}$	law of MAP with discrete modulator	287
$\mathbf{P}_{x,\theta}$	law of MAP with continuous modulator	300
$\boldsymbol{Q}$	intensity matrix of the discrete modulator	289
$\boldsymbol{G}$	matrix of Laplace transforms of inter-modulator jumps	289
$\boldsymbol{\pi}$	stationary distribution of $\boldsymbol{Q}$	289
$\boldsymbol{\Delta_\pi}$	diagonal matrix populated with $\boldsymbol{\pi}$	290
$\boldsymbol{\Psi},\ \hat{\boldsymbol{\Psi}}$	matrix exponent of MAP and MAP dual	289, 290
$\overline{m}_t$	time at which the MAP ordinate last visits its past maximum before time t	295
(H^+, J^+)	ascending ladder MAP	293
$\boldsymbol{\kappa}(\gamma, \lambda)$	matrix of exponent of space-time ascending ladder MAP	294
$\Phi_i(\gamma, \lambda)$	space-time Laplace exponents of pure subordinator states of ascending ladder MAP	294
$\chi(z),\ \boldsymbol{v}(z)$	eigenvalue and right eigenvector of matrix exponent of $\boldsymbol{\Psi}(z)$	291
$\boldsymbol{\kappa}(\lambda)$	matrix exponent of the ascending ladder MAP	299
$\boldsymbol{U}_{i,j}(x)$	ascending and descending ladder MAP resolvent for discrete modulator	335
$\mathbf{R}_z[f](\theta)$	MAP resolvent	403
$\boldsymbol{\rho}_z[f](\theta)$, $\hat{\boldsymbol{\rho}}_z[f](\theta)$	ascending and descending ladder MAP resolvent for continuous modulator	403

Self-similar Markov processes

I_t and I_∞	integrated exponential Lévy process underlying a self-similar Markov process	116
$\varphi(t)$	right inverse of integrated exponential Lévy process	116
(Z, P)	self-similar Markov process	116, 227
ζ	lifetime of process	115
$(\Xi, \mathbf{P})$	Lévy process underlying a positive self-similar Markov process	116, 227
$\mathtt{D}_x$	last passage time below x	120
$\overleftarrow{Z}_t$	time reversed process from last passage time	228
$\overleftarrow{S}_y$	first passage time below y of reversed pssMp	228
S_y	first passage time of pssMp above y	249
Γ_1	left limit of positive self-similar Markov process at last passage at level x_1	229

$\mathtt{J}_t$	future infimum of pssMp after time t	240
Υ	scaled left limit of positive self-similar Markov process at last passage at any level	240
$\tilde{p}$	density of distribution of $\Upsilon^\delta I(\delta, \Xi)$	255
F, F_q	right tail distribution of integrated exponential and partially integrated exponential dual Lévy process	234
$\overline{F}_\Upsilon$	left tail distribution of $\Upsilon^\alpha \hat{I}_\infty$	241
G	left tail distribution of S_1 under P_0	248

Excursions

$L,\ \ell$	local time	44, 178, 383
ϵ_t	canonical excursion at local time t	45, 178
$(\epsilon, J^\epsilon),\ (\epsilon, \Theta^\epsilon)$	canonical radial excursion at local time t	294, 384
ζ	excursion lifetime	45
$\overline{\mathcal{U}}(\mathbb{R}), \underline{\mathcal{U}}(\mathbb{R})$	space of excursion paths of Lévy processes from maximum and minimum	45
$\overline{\mathcal{U}}(\mathbb{R} \times E)$	space of MAP excursions from ordinate maximum	294
$\underline{\mathcal{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$	space of MAP excursions from ordinate minimum	385
$\overline{n}$ resp. $\underline{n}$	excursion measure of a Lévy process from its maximum resp. minimum	45, 177
n_i	MAP excursion measure when left end point begins with modulator in state i	294
$\mathbb{N}_\theta$	radial excursion measure when left end point begins with modulator in state θ	384

Other notation

$\mathbf{e}_q$	independent and exponentially distributed random variable	30
$\Gamma(z)$	gamma function	431
${}_2F_1(a, b; c; z)$	hypergeometric function	435
$G(z; \tau)$	double gamma function	95, 432
$S_2(z; \tau)$	double sine function	185

$F(s)$	special function derived from double gamma functions	96
C_0, C_∞	families of positive increasing functions with growth at 0 and ∞, respectively	241, 248
$\mathbb{S}^{d-1}(b, r)$	sphere in $\mathbb{R}^d$ of radius r, centred at b	352
$\sigma_a(\mathrm{d}z)$	surface measure on $\mathbb{S}^{d-1}(0, a)$ normalised to have unit mass	356
$K(x)$	spatial inversion through unit sphere (Kelvin transform)	324
$x^*,\ x^\diamond$	non-centred sphere inversion and non-centred sphere inversion with reflection	352, 354

Preface

There have been a number of developments in the theory of α-stable Lévy processes in recent years. This is largely thanks to a better understanding of their connection to self-similar Markov processes in conjunction with a revised view on the complex analysis that can subsequently be brought into play. We mention in this respect the paper of Caballero and Chaumont [43] as well as the work of Kuznetsov [115, 116], both of which present seminal perspectives in terms of the underlying Wiener–Hopf theory that has stimulated a large base of literature. Among this literature, the PhD theses of Alex Watson in 2013 and Weerapat Satitkanitkul in 2018 stand out.

The basic idea of this book is to give an introductory account of these developments and, accordingly, expose the new techniques that have appeared in the literature since the mid-2000s. The majority of the mathematical computations that are developed in the following chapters pertain either to recent material or to a new approach for classical results. At the end of each chapter, a section is devoted to referencing all material presented in the main body of the chapter. An appendix is also included, and referred to throughout the text, to record some of the more specialist facts from complex analysis, special functions and the theory of Markov processes that are used in the text.

We hope that this text will serve as a standard reference for those interested in the modern theory of α-stable Lévy processes as well as suitable material for a graduate course. Indeed, some of the material in this text has been used in conjunction with lectures given by AEK at the University of Zurich, the National Technical University of Athens, University of Jyväskylä, The Chinese Academy of Sciences and at Prob-L@B in Bath, as well as by JCP at UNAM in Mexico City, CIMAT in Guanajuato and Kyoto University.

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