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Stable distributions

The starting point of this monograph is the notion of distributional stability and infinite divisibility. Stable distributions are the celebrated class which exhibit both of the aforesaid properties and, accordingly, offer a number of remarkably explicit formulae and identities. We therefore begin our journey by addressing the robust mathematical theory that supports the characterisation of stable distributions in preparation for later chapters.

1.1 One-dimensional stable distributions

We begin our discussion by first restricting ourselves to the one-dimensional setting. The following definition, for which we use $\stackrel{(d)}{=}$ to mean equality in distribution, is key to the notion of distributional stability.

Definition 1.1 A non-degenerate random variable X has a *stable* distribution if, for any $a > 0$ and $b > 0$, there exists $c > 0$, such that

$$aX_1 + bX_2 \stackrel{(d)}{=} cX, \tag{1.1}$$

where X_1 and X_2 are independent and $X_1 \stackrel{(d)}{=} X_2 \stackrel{(d)}{=} X$. We exclude from this definition the possibility that $X \equiv 0$.

The experienced reader will immediately spot that Definition 1.1 pertains to what is more broadly known in the literature as a *strictly stable* random variable. The notion of a *stable* random variable is reserved for a slightly broader concept. Since we will never have occasion in this book to distinguish the difference, we will depart from the traditional convention and refer only to stable random variables, using this definition.

Observe that (1.1) implies

$$X \overset{(d)}{=} \frac{X_1 + X_2}{d_2},$$

for some constant $d_2 > 0$. By induction, it is easy to see that, for any $n \geq 0$, there exists a constant $d_n > 0$ and n independent random variables X_i, $1 \leq i \leq n$, with the same distribution as X, such that

$$X \overset{(d)}{=} \frac{X_1 + X_2 + \cdots + X_n}{d_n}. \tag{1.2}$$

Said another way, any stable random variable X is infinitely divisible.

For convenience, let us recall the so-called Lévy–Khintchine representation, which provides a complete characterisation of infinitely divisible distributions. We first introduce some notation. Let μ be the probability distribution of a real-valued random variable and define its characteristic function by

$$\hat{\mu}(z) = \int_{\mathbb{R}} e^{izx}\,\mu(dx), \qquad z \in \mathbb{R}.$$

If μ is an infinitely divisible distribution, then it is known that its characteristic function never vanishes. As a consequence, there exists a continuous function $\Psi\colon \mathbb{R} \mapsto \mathbb{C}$, called the *characteristic exponent* of μ, such that $\Psi(0) = 0$, and

$$\exp\{-\Psi(z)\} := \hat{\mu}(z), \quad \text{for } z \in \mathbb{R}. \tag{1.3}$$

Theorem 1.2 (Lévy–Khintchine representation) *A function $\Psi\colon \mathbb{R} \mapsto \mathbb{C}$ is the characteristic exponent of an infinitely divisible random variable if and only if there exists a triple (a, σ, Π), where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$, such that*

$$\Psi(z) = iaz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(1 - e^{izx} + izx\mathbf{1}_{(|x|<1)}\right) \Pi(dx), \tag{1.4}$$

for every $z \in \mathbb{R}$. Moreover, the triple (a, σ^2, Π) is unique within the given arrangement on the right-hand side of (1.4).

The measure Π is called *the Lévy measure* of the distribution μ and σ its *Gaussian coefficient*. Whilst the triple (a, σ, Π) defining $\Psi(z)$ is unique as described, in various situations one may prefer to use a different *regularising* function $h(x)$, in which case (1.4) is written as

$$\Psi(z) = i\tilde{a}z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(1 - e^{izx} + izh(x)\right) \Pi(dx), \qquad z \in \mathbb{R}, \tag{1.5}$$

where

$$\tilde{a} = a - \int_{\mathbb{R}} (h(x) - x\mathbf{1}_{(|x|<1)})\,\Pi(\mathrm{d}x),$$

which is finite.

In this chapter, we shall interchange between the two equivalent representations given by (1.4) and (1.5). For example, when the measure Π satisfies the stronger condition

$$\int_{\mathbb{R}} (1 \wedge |x|)\,\Pi(\mathrm{d}x) < \infty,$$

we may choose $h(x) \equiv 0$. If the distribution μ has finite mean, we may choose $h(x) \equiv x$. In some cases, it will be convenient to choose $h(x) = \sin(x)$ or $h(x) = x/(1+x^2)$. Everywhere in this book, when we say that the distribution μ has characteristic triple (a, σ, Π) without specifying the regularising function h, we assume that the characteristic exponent is given via (1.4), otherwise we will say that the distribution μ has characteristic triple (a, σ, Π) with the regularising function h, in which case Ψ will be given by (1.5).

The following main result provides the explicit characteristic exponent of stable distributions. As part of its proof, which will be provided in the next section, we will also be obliged to understand the structure of the underlying triple (a, σ, Π) in the associated Lévy–Khintchine formula.

Theorem 1.3 *A stable random variable X has a characteristic exponent satisfying*

$$\Psi(z) = c|z|^{\alpha}\Big(1 - \mathrm{i}\beta \tan(\pi\alpha/2)\,\mathrm{sgn}(z)\Big), \qquad z \in \mathbb{R}, \tag{1.6}$$

where

$$\alpha \in (0,1) \cup (1,2],\ c > 0 \quad \text{and} \quad \beta \in [-1,1]$$

or

$$\alpha = 1, \beta = 0 \quad \text{and we understand} \quad \beta \tan(\pi\alpha/2) := 0.$$

The latter case is known as the symmetric Cauchy distribution.

Remark 1.4 Note that the symmetric Cauchy distribution with drift $\delta \in \mathbb{R}$, that is,

$$\Psi(z) = c|z| + \delta z, \qquad z \in \mathbb{R},$$

also belongs to the class of one-dimensional stable distributions. Nonetheless, we will henceforth only deal with the case that $\delta = 0$ when $\alpha = 1$.

Remark 1.5 We also note that the case $\alpha = 2$ corresponds to the case where X has a Gaussian distribution. As we shall see in Chapter 2, associated to each of the distributions discussed in this chapter is a Lévy process. As one might expect, the case $\alpha = 2$ leads to Brownian motion. For other values of α, we will find an association with Lévy processes that do not have continuous paths, the so-called α-stable processes (also referred to as just stable processes). It is the case of processes with path discontinuities that forms the primary concern of this book. For this reason, the overwhelming majority of this text will be restricted to the setting that $\alpha \in (0, 2)$.

1.2 Characteristic exponent of a one-dimensional stable law

We dedicate this section entirely to the proof of Theorem 1.3. As part of this process, we need to establish two key intermediary results.

Lemma 1.6 *The sequence $(d_k)_{k\geq 1}$ defined by (1.2) is strictly increasing and satisfies $d_k = k^{1/\alpha}$ for some $\alpha > 0$, $k \geq 1$.*

Proof Recall that $\hat{\mu}$ denotes the characteristic function of a stable distribution X and, thanks to the infinite divisibility of X, $\hat{\mu}(z) \neq 0$ for $z \in \mathbb{R}$. From the definition of the sequence $(d_k)_{k\geq 1}$, the scaling property in (1.2) can be reworded to say

$$\Psi(d_k z) = k\Psi(z), \qquad z \in \mathbb{R}, \;\; k \geq 1. \tag{1.7}$$

In turn, this implies $|\hat{\mu}(d_{k+1}z)| = |\hat{\mu}(z)||\hat{\mu}(d_k z)| \leq |\hat{\mu}(d_k z)|$ and hence

$$\left|\hat{\mu}\left(\frac{d_{k+1}}{d_k} z\right)\right| \leq |\hat{\mu}(z)|, \qquad k \geq 1.$$

We are now forced to conclude that $d_{k+1} \geq d_k$, for $k \geq 1$. To see why, note that

$$\left|\hat{\mu}\left(\left(\frac{d_{k+1}}{d_k}\right)^n z\right)\right| \leq |\hat{\mu}(z)|, \qquad \text{for any} \quad n \geq 1,$$

with $(d_{k+1}/d_k)^n \to 0$ as $n \to \infty$, which would imply that $1 \leq |\hat{\mu}(z)|$, leading to a contradiction.

Next, we observe that for all $m, n \geq 1$ and $z \in \mathbb{R}$,

$$|\Psi(d_{mn}z)| = mn|\Psi(z)| = n\Big|m\Psi(z)\Big| = \Big|n\Psi(d_m z)\Big| = |\Psi(d_n d_m z)|,$$

implying that $d_{mn} = d_n d_m$. In particular, for any positive integer j, $d_{m^j} = d_m^j$. If $1 < n < m$, there is a positive integer p such that $m^j \leq n^p < m^{j+1}$. Using these inequalities and the established monotonicity of $(d_k)_{k\geq 1}$, we have

$$\frac{j}{j+1}\frac{\log d_m}{\log m} \le \frac{\log d_n}{\log n} \le \frac{j+1}{j}\frac{\log d_m}{\log m}.$$

Hence, taking $j \to \infty$, we get

$$\frac{\log d_m}{\log m} = \frac{\log d_n}{\log n} =: \frac{1}{\alpha},$$

for some strictly positive constant α. Therefore $\log d_n = \log n^{1/\alpha}$ or equivalently $d_n = n^{1/\alpha}$, for $n \ge 1$ and $\alpha > 0$. □

Our second intermediary result characterises the form of the underlying Lévy measure of any stable distribution.

Proposition 1.7 *If X is a stable random variable, then necessarily $\alpha \in (0, 2]$. In the case that $\alpha = 2$, X is Gaussian distributed. Otherwise when $\alpha \in (0, 2)$, then there exist $c_1, c_2 \ge 0$ such that $c_1 + c_2 > 0$ and the underlying Lévy measure Π satisfies*

$$\Pi(\mathrm{d}x) = |x|^{-1-\alpha}\Big(c_1\mathbf{1}_{(x>0)} + c_2\mathbf{1}_{(x<0)}\Big)\,\mathrm{d}x, \qquad x \in \mathbb{R}. \tag{1.8}$$

Proof Recall that identity (1.7) and Lemma 1.6 imply $k\Psi(z) = \Psi(k^{1/\alpha}z)$, for $z \in \mathbb{R}$ and $k \ge 1$. More precisely, we observe

$$\begin{aligned}&\mathrm{i}kaz + \frac{1}{2}k\sigma^2 z^2 + \int_{\mathbb{R}}\Big(1 - \mathrm{e}^{\mathrm{i}zx} + \mathrm{i}zx\mathbf{1}_{(|x|<1)}\Big)\,k\,\Pi(\mathrm{d}x)\\ &\quad = \mathrm{i}ak^{1/\alpha}z + \frac{1}{2}\sigma^2 z^2 k^{2/\alpha} + \int_{\mathbb{R}}\Big(1 - \mathrm{e}^{\mathrm{i}zk^{1/\alpha}x} + \mathrm{i}zk^{1/\alpha}x\mathbf{1}_{(|x|<1)}\Big)\,\Pi(\mathrm{d}x),\end{aligned} \tag{1.9}$$

for any $k \ge 1$ and $z \in \mathbb{R}$. Hence if $\sigma > 0$, we are forced to take $\alpha = 2$. Moreover, still in the setting $\alpha = 2$, if we then let k tend to ∞, the latter identity implies $a = 0$ and $\Pi \equiv 0$. In conclusion, the case that $\alpha = 2$ corresponds to a Gaussian random variable.

Next, we assume $\sigma = 0$. Again from identity (1.9), by changing variables in the integral on the right-hand side, we deduce

$$k\Pi(\mathrm{d}x) = \Pi(k^{-1/\alpha}\mathrm{d}x), \qquad x \neq 0.$$

Therefore, for the functions $\overline{\Pi}^{(+)}(x) := \Pi([x, \infty))$, $x > 0$, and $\overline{\Pi}^{(-)}(x) := \Pi((-\infty, x))$, $x < 0$, we have

$$\overline{\Pi}^{(+)}(x) = \frac{1}{k}\overline{\Pi}^{(+)}\left(k^{-1/\alpha}x\right) \qquad \text{and} \qquad \overline{\Pi}^{(-)}(x) = \frac{1}{k}\overline{\Pi}^{(-)}\left(k^{-1/\alpha}x\right).$$

From the first of these two, we have, for all $k, n \ge 1$,

$$\frac{1}{n}\overline{\Pi}^{(+)}\left(\frac{k^{1/\alpha}}{n^{1/\alpha}}\right) = \overline{\Pi}^{(+)}\left(k^{1/\alpha}\right) = \frac{1}{k}\overline{\Pi}^{(+)}(1).$$

Since $\{(k/n)^{1/\alpha}; k, n \in \mathbb{N}\}$ is dense in $[0, \infty)$ and the function $\overline{\Pi}^{(+)}$ is non-increasing, we deduce $\overline{\Pi}^{(+)}(x) = x^{-\alpha}\overline{\Pi}^{(+)}(1)$, for $x > 0$. Similarly, we may deduce $\overline{\Pi}^{(-)}(x) = |x|^{-\alpha}\overline{\Pi}^{(-)}(1)$, for $x < 0$.

Now taking $c_1 := \alpha\overline{\Pi}^{(+)}(1)$ and $c_2 := \alpha\overline{\Pi}^{(-)}(1)$, we obtain

$$\Pi(\mathrm{d}x) = |x|^{-1-\alpha}\Big(c_1\mathbf{1}_{(x>0)} + c_2\mathbf{1}_{(x<0)}\Big)\,\mathrm{d}x, \qquad x \in \mathbb{R},$$

as required. As Π is a Lévy measure, in particular, it must satisfy the integral condition

$$\int_{\mathbb{R}} (1 \wedge |x|^2)\,\Pi(\mathrm{d}x) < \infty.$$

We thus deduce that $\alpha \in (0, 2)$. □

Finally, we are ready to compute the characteristic exponent Ψ as stated in Theorem 1.3.

Proof of Theorem 1.3 Since the case $\alpha = 2$ has already been characterised as Gaussian in the proof of Proposition 1.7, we set $\sigma = 0$ and focus on the case $\alpha \in (0, 2)$.

We first observe that, when $\alpha \in (0, 1)$ the function $x \mapsto |x|^{-(\alpha+1)}$ is integrable near 0 and hence we may take the regularising function in (1.5) to satisfy $h(x) = 0$. From identity (1.7), we deduce that $\tilde{a} = 0$ in (1.5), or in other words,

$$a = -\int_{(|x|<1)} x\,\Pi(\mathrm{d}x).$$

Using the well-known integral identity for the gamma function, see for instance (A.7) in the Appendix, we have

$$\int_0^\infty \mathrm{e}^{\mathrm{i}zx} x^{s-1}\,\mathrm{d}x = z^{-s}\Gamma(s)\mathrm{e}^{\pi\mathrm{i}s/2}, \quad z > 0,\ 0 < s < 1, \tag{1.10}$$

and, appealing to integration by parts, we find that

$$\int_0^\infty \Big(\mathrm{e}^{\mathrm{i}zx} - 1\Big)\,x^{-1-\alpha}\,\mathrm{d}x = z^\alpha\mathrm{e}^{-\pi\mathrm{i}\alpha/2}\Gamma(-\alpha), \qquad z > 0. \tag{1.11}$$

Making the change of variable $x \mapsto -x$ and taking the complex conjugate of both sides we find

$$\int_{-\infty}^0 \Big(\mathrm{e}^{\mathrm{i}zx} - 1\Big)\,|x|^{-1-\alpha}\,\mathrm{d}x = \int_0^\infty \Big(\mathrm{e}^{-\mathrm{i}zx} - 1\Big)\,x^{-1-\alpha}\,\mathrm{d}x = z^\alpha\mathrm{e}^{\pi\mathrm{i}\alpha/2}\Gamma(-\alpha), \tag{1.12}$$

for $z > 0$. Note also that when z takes negative values, we can similarly make use of the computations leading to (1.12). Then, we apply the following simple identity

$$c_1 e^{-\pi i\alpha/2} + c_2 e^{\pi i\alpha/2} = (c_1 + c_2)\cos(\pi\alpha/2)\left(1 - i\frac{c_1 - c_2}{c_1 + c_2}\tan(\pi\alpha/2)\right),$$

and observe that

$$c = -(c_1 + c_2)\Gamma(-\alpha)\cos(\pi\alpha/2) > 0,$$

since $-\Gamma(-\alpha)$ is positive for $\alpha \in (0,1)$. This completes the proof of the case $\alpha \in (0,1)$.

When $\alpha \in (1,2)$, the function $x \mapsto |x|^{-(\alpha+1)}$ integrates x^2 in a neighbourhood of 0 and hence we may take the regularising function in (1.5) as $h(x) = x$. Again identity (1.7) implies $\tilde{a} = 0$ in (1.5), and therefore

$$a = \int_{(|x|\geq 1)} x\,\Pi(\mathrm{d}x).$$

Similarly, we use (1.10) and apply integration by parts twice to find

$$\int_0^\infty \left(e^{izx} - 1 - izx\right) x^{-1-\alpha}\,\mathrm{d}x = z^\alpha e^{-\pi i\alpha/2}\Gamma(-\alpha), \tag{1.13}$$

for $z > 0$, and the rest of the proof proceeds in the same way as in the case $\alpha \in (0,1)$.

Finally, the case $\alpha = 1$ must be treated differently. In this case, we observe

$$\int_0^\infty \left(1 - e^{izx} + izx\mathbf{1}_{(|x|<1)}\right)\frac{\mathrm{d}x}{x^2} = \int_0^\infty (1 - \cos(zx))\frac{\mathrm{d}x}{x^2} - i\int_0^\infty \left(\sin(zx) - zx\mathbf{1}_{(|x|<1)}\right)\frac{\mathrm{d}x}{x^2}. \tag{1.14}$$

A change of variables followed by integration by parts gives us

$$\int_0^\infty (1 - \cos(zx))\frac{\mathrm{d}x}{x^2} = |z|\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x = |z|\int_0^\infty\int_0^\infty \sin(x)e^{-xu}\,\mathrm{d}u\,\mathrm{d}x.$$

Since

$$\int_0^\infty e^{-xu}\sin(x)\,\mathrm{d}x = \frac{1}{u^2+1}, \tag{1.15}$$

we get

$$\int_0^\infty (1 - \cos(zx))\frac{\mathrm{d}x}{x^2} = \frac{|z|\pi}{2}. \tag{1.16}$$

Next, for simplicity, we assume that $z > 0$. Observe that

$$\begin{aligned}\int_0^\infty (\sin(zx) - zx\mathbf{1}_{(|x|<1)})\,\frac{\mathrm{d}x}{x^2} &= \int_0^{1/z} (\sin(zx) - zx)\,\frac{\mathrm{d}x}{x^2} \\ &\quad + \int_{1/z}^\infty \sin(zx)\frac{\mathrm{d}x}{x^2} - z\log z \\ &= z\left(\int_0^1 (\sin(x) - x)\,\frac{\mathrm{d}x}{x^2} + \int_1^\infty \sin(x)\,\frac{\mathrm{d}x}{x^2}\right) - z\log z. \qquad (1.17)\end{aligned}$$

Hence by defining

$$K := \int_0^1 (\sin(x) - x)\,\frac{\mathrm{d}x}{x^2} + \int_1^\infty \sin(x)\,\frac{\mathrm{d}x}{x^2},$$

and putting all the pieces in (1.16) and (1.17) back into (1.14), we deduce

$$\int_0^\infty \left(1 - \mathrm{e}^{\mathrm{i}zx} + \mathrm{i}zx\mathbf{1}_{(|x|<1)}\right)\frac{\mathrm{d}x}{x^2} = \frac{|z|\pi}{2} - \mathrm{i}Kz + \mathrm{i}z\log|z|, \quad z \in \mathbb{R}\setminus\{0\}.$$

Therefore, from Proposition 1.7 and the above reasoning, the characteristic exponent Ψ satisfies

$$\Psi(z) = iaz + (c_1 - c_2)\mathrm{i}Kz + (c_1 + c_2)|z|\frac{\pi}{2} + (c_1 - c_2)\mathrm{i}z\log|z|, \quad z \in \mathbb{R}\setminus\{0\}.$$

As we must have $\Psi(k^{1/\alpha}z) = k\Psi(z)$, $z \in \mathbb{R}$, $k \in \mathbb{N}$, albeit now $\alpha = 1$, from Lemma 1.6, we deduce that $c_1 = c_2$ and then

$$\Psi(z) = \mathrm{i}az + (c_1 + c_2)|z|\frac{\pi}{2}, \quad z \in \mathbb{R}.$$

Taking note of Remark 1.4, by taking $a = 0$, we get the desired result. □

Reviewing the proof here, we also get some information about the constants c_1 and c_2, appearing in Proposition 1.7, in relation to the parameters c and β in (1.6).

Corollary 1.8 *When $\alpha \in (0, 2)$, the constants c_1, c_2 appearing in the Lévy measure* (1.8) *satisfy*

$$c = -(c_1 + c_2)\Gamma(-\alpha)\cos(\pi\alpha/2) \quad \textit{and} \quad \beta = \frac{c_1 - c_2}{c_1 + c_2}, \qquad (1.18)$$

when $\alpha \in (0, 1) \cup (1, 2)$. Moreover, $c_1 = c_2$ with $c = c_1\pi$, when $\alpha = 1$.

We also get from the proof of Theorem 1.3 the values of a in the Lévy–Khintchine triple 1.4. As such, the following corollary completes the statement of Proposition 1.7.

Corollary 1.9 *When $\alpha \in (0,1)$, the constant a in the Lévy–Khintchine triple is equal to $-\int_{(|x|<1)} x\Pi(\mathrm{d}x)$, when $\alpha \in (1,2)$, we have $a = \int_{(|x|\geq 1)} x\Pi(\mathrm{d}x)$ and when $\alpha = 1$, we have $a = 0$.*

1.3 Moments

An important feature of stable distributions when $\alpha \in (0,2)$, which is one of their signature properties that differs from the setting that $\alpha = 2$, is that they do not possess second moments (and hence no other greater moments). The precise cut-off where positive moments exist is the concern of the next main result.

Theorem 1.10 *Suppose that X is a stable distribution with index $\alpha \in (0,2)$. Then $\mathbb{E}[|X|^\beta] < \infty$, for $0 \leq \beta < \alpha$, and for $\beta \geq \alpha$, we have $\mathbb{E}[|X|^\beta] = \infty$.*

Proof We start by noting that, irrespective of the symmetry in the distribution of X, thanks to the shape of Π given in Theorem 1.7, we have

$$\int_{(|x|\geq 1)} |x|^\beta \, \Pi\,(\mathrm{d}x) < \infty,$$

for $\beta \in [0,\alpha)$ and infinite for $\beta \in [\alpha,\infty)$.

Next, note that the Lévy–Khintchine exponent (1.4), written here as Ψ, has $\sigma = 0$ and can be decomposed in the form $\Psi = \Psi^{(1)} + \Psi^{(2)}$, where

$$\Psi^{(1)}(z) = \mathrm{i}az + \int_{(|x|\geq 1)} \left(1 - \mathrm{e}^{\mathrm{i}zx}\right) \Pi(\mathrm{d}x), \qquad z \in \mathbb{R},$$

and

$$\Psi^{(2)}(z) = \int_{(|x|<1)} \left(1 - \mathrm{e}^{\mathrm{i}zx} + \mathrm{i}zx\right) \Pi(\mathrm{d}x), \qquad z \in \mathbb{R},$$

with

$$a = \begin{cases} -\int_{(|x|<1)} x\Pi(\mathrm{d}x) & \text{if } \alpha \in (0,1), \\ 0 & \text{if } \alpha = 1, \\ \int_{(|x|\geq 1)} x\Pi(\mathrm{d}x) & \text{if } \alpha \in (1,2). \end{cases}$$

For the first of these two, we note that it corresponds to the characteristic exponent of a compound Poisson random variable, say

$$X^{(1)} = -a + \sum_{i=1}^{\mathtt{N}} \Xi_i,$$

where $\mathtt{N}$ is an independent Poisson distributed random variable with rate $\Pi(|x| \geq 1)$ and $(\Xi_i, i \geq 1)$ are i.i.d. with distribution $\Pi(|x| \geq 1)^{-1}\Pi(\mathrm{d}x)\mathbf{1}_{(|x|\geq 1)}$.

(We use the usual convention that $\sum_{i=1}^{0} := 0$.) We want to consider the moments of $X^{(1)}$. It is already clear from the tail of Π that Ξ_1 has a finite β-moment if $\beta \in [0,\alpha)$ and infinite β-moment if $\beta \geq \alpha$. In particular, Ξ_1 has a first moment (and hence all smaller positive moments) if and only if $\alpha \in (1,2)$.

When Ξ_1 has a first moment, that is, $\alpha \in (1,2)$, we observe that $X^{(1)}$ can be rewritten as

$$X^{(1)} = \sum_{i=1}^{\mathtt{N}} \widetilde{\Xi}_i,$$

where each of the $\widetilde{\Xi}_i$ has zero mean. In that case, we may appeal to an inequality for martingale differences, which states that, for $\beta \in [1,\alpha)$ and $n \geq 1$,

$$\mathbb{E}\Big[\Big|\sum_{i=1}^{n} \widetilde{\Xi}_i\Big|^{\beta}\Big] \leq 2^{\beta} \sum_{i=1}^{n} \mathbb{E}[|\widetilde{\Xi}_i|^{\beta}]. \tag{1.19}$$

As the right-hand side is equal to $2^{\beta} n \mathbb{E}[|\widetilde{\Xi}_1|^{\beta}]$, it follows by an independent randomisation of n by the Poisson distribution of $\mathtt{N}$ that $\mathbb{E}[|X^{(1)}|^{\beta}] < \infty$.

When $X^{(1)}$ has no first moment, that is, $\alpha \in (0,1]$, we can use the inequality

$$\Big(\sum_{i=1}^{n} u_i\Big)^q \leq \sum_{i=1}^{n} u_i^q, \qquad u_1,\ldots,u_n \geq 0, \tag{1.20}$$

for $q \in (0,1]$, to deduce that

$$\mathbb{E}\Big[\Big|\sum_{i=1}^{n} \Xi_i\Big|^{\beta}\Big] \leq \mathbb{E}\Big[\Big(\sum_{i=1}^{n} |\Xi_i|\Big)^{\beta}\Big] \leq \sum_{i=1}^{n} \mathbb{E}\left[|\Xi_i|^{\beta}\right] = n\mathbb{E}\left[|\Xi_1|^{\beta}\right] < \infty,$$

for $\beta \in [0,\alpha)$. Hence, again following an independent randomisation of n by the distribution of $\mathtt{N}$, $\mathbb{E}[|X^{(1)}|^{\beta}] < \infty$, for $\beta \in [0,\alpha)$.

Next, we want to show that $\mathbb{E}[|X^{(2)}|^{\beta}] < \infty$, for $\beta \in [0,\alpha)$ and $\alpha \in (0,2)$, where $X^{(2)}$ is the random variable whose characteristic exponent is given by $\Psi^{(2)}$. To this end, we write

$$\Psi^{(2)}(z) = -\int_{(|x|<1)} \sum_{k\geq 0} \frac{(\mathrm{i}zx)^{k+2}}{(k+2)!} \Pi(\mathrm{d}x). \tag{1.21}$$

The sum and the integral may be exchanged using Fubini's Theorem and the estimate

$$\sum_{k\geq 0} \int_{(|x|<1)} \frac{|zx|^{k+2}}{(k+2)!} \Pi(\mathrm{d}x) \leq \sum_{k\geq 0} \frac{|z|^{k+2}}{(k+2)!} \int_{(|x|<1)} x^2 \Pi(\mathrm{d}x) < \infty.$$

Hence, the right-hand side of (1.21) can be written as a power series for all $z \in \mathbb{C}$ and is thus entire. In turn this guarantees that $\hat{\mu}^{(2)}(z) := \exp\{-\Psi^{(2)}(z)\}$