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## FROM CATEGORIES TO HOMOTOPY THEORY

Category theory provides structure for the mathematical world and is seen everywhere in modern mathematics. With this book, the author bridges the gap between pure category theory and its numerous applications in homotopy theory, providing the necessary background information to make the subject accessible to graduate students or researchers with a background in algebraic topology and algebra.

The reader is first introduced to category theory, starting with basic definitions and concepts, before progressing to more advanced themes. Concrete examples and exercises illustrate the topics, ranging from colimits to constructions such as the Day convolution product.

Part II covers important applications of category theory, giving a thorough introduction to simplicial objects including an account of quasi-categories and Segal sets. Diagram categories play a central role throughout the book, giving rise to models of iterated loop spaces, and feature prominently in functor homology and homology of small categories.

**Birgit Richter** is Professor of Mathematics at the University of Hamburg. She is the co-editor of *Structured Ring Spectra* (2004) and *New Topological Contexts for Galois Theory and Algebraic Geometry* (2009).

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# From Categories to Homotopy Theory

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To my father

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