CAMBRIDGE
Cambridge University Press
978-1-108-47962-2 — From Categories to Homotopy Theory
Birgit Richter
Excerpt
More Information

Introduction

Category theory has at least two important features. The first one is that it allows us to structure our mathematical world. Many constructions that you encounter in your daily life look structurally very similar, like products of sets, products of topological spaces, and products of modules, and then you might be delighted to learn that there is a notion of a product of objects in a category and all the above examples are actually just instances of such products, here in the category of sets, topological spaces, and modules, so you don't have to reprove all the properties products have, because they hold for every such construction. So category theory helps you to recognize things as what they are.

It also allows you to express objects in a category by something that looks apparently way larger. For instance, the Yoneda lemma describes a set of the form $F(C)$ (where C is an object of some category and F is a functor from that category to the category of sets) as the set of natural transformations between another nice functor and F. This might look like a bad deal, but in this set of natural transformations you can manipulate things and this reinterpretation for instance gives you cohomology operations as morphisms between the representing objects.

Another feature is that you can actually use category theory in order to build topological spaces and to do homotopy theory. A central example is the nerve of a (small) category: You view the objects of your category as points, every morphism gives a 1-simplex, a pair of composable morphisms gives a 2-simplex, and so on. Then you build a topological space out of this by associating a topological n-simplex to an n-simplex in the nerve, but you do some nontrivial gluing, for instance, identity morphisms don't really give you any information so you shrink the associated edges. In the end you get a CW complex $B\mathcal{C}$ for every small category $\mathcal{C}$. Properties of categories and functors translate into properties of this space and continuous maps between such spaces. For instance, a natural transformation between two functors gives rise to a homotopy between the induced maps, and an equivalence of categories gives a homotopy equivalence of the corresponding classifying spaces.

Classifying spaces of categories give rise to classifying spaces of groups but you can also use them and related constructions to build the spaces of the algebraic K-theory spectrum of a ring, you can give models for iterated based loop spaces and you can construct explicit models of homotopy colimits and much more.

This book has two parts. The first one gives an introduction to category theory describing its basic definitions and constructions, so this part focuses on the first feature of category theory. The second part presents applications to homotopy theory. Here, "homotopy theory" does not mean any precisely confined area of algebraic topology but rather some collection of topics that is heavily influenced by my research interests and my personal taste. An emphasis is on simplicial methods, on functor categories, on some concepts that are crucial for algebraic K-theory, on models for iterated based loop spaces, and on applications to homological algebra. The book also contains an account of functor homology and of homology of small categories. These are two concepts that in my opinion deserve to be wider known by working mathematicians, in particular, by working algebraic topologists. Many prominent examples of homology theories can be expressed that way, and even if you are not interested in homology theories per se, you might stumble across a spectral sequence whose E^2 term happens to consist of such homology groups – and then it might be helpful to recognize these groups, because then you have other means of understanding and calculating them.

One thing that you might realize is that I love diagrams. If I see a proof that uses a lengthy reformulation for showing that one thing (functor, natural transformation, etc.) is the same as a second thing, then I usually don't understand such a proof before I "translate" it into a diagram that has to commute, so I usually end up drawing the corresponding diagram. My hope is that this approach isn't just helpful for me. So in a lot of places in this book you will find proofs that more or less just consist of showing that a certain diagram commutes by displaying the diagram and dissecting its parts. I also love examples and therefore there are plenty of examples in the book. There are also some exercises. These are not meant to be challenging, but they want to nudge you to actually learn how to work with the concepts that are introduced and how to deal with examples. One danger in category theory is that one learns the abstract theory, and then, if confronted with an example, one doesn't really know what to do. I hope that the examples and exercises help to avoid this.

My hope is that this book will bridge a gap: There are several very good accounts on category theory, for instance, [**Rie16, Bo94-1, Bo94-2, ML98, Sch70**], and there are many excellent sources on applications of categorical methods to topics such as algebraic K-theory [**Q73, Gr76, DGM13, W13**], the theory of iterated loop spaces, models for categories of spectra and many more, but in the latter texts it is assumed that the reader is familiar with the relevant concepts from category theory and sometimes it can be difficult to collect the necessary background. This is not a book on ∞-categories, but I cover quasi-categories, joins, slices, cocartesian fibrations, and the category Θ_n and some other things related to ∞-categories, and these might help you with digesting Jacob Lurie's books [**Lu09, Lu∞**] and other sources. For an overview on quasi-categories and some of their applications, I recommend Moritz Groth's survey [**Gro20**]. For a comparison between different models of ∞-categories, Julie Bergner's book [**B18**] is an excellent source. My book is also not an introduction to model categories, but if you want to dive deeper into some of the applications and you decide to read the papers that I mention, then you will need them. There are very good sources for learning about model categories, and in increasing level of complexity I recommend [**DwSp95, Ho99, Hi03**], but of course also the original account [**Q67**].

In my opinion (many people would disagree), you should not learn category theory before you have seen enough examples of categories in your mathematical life, so before you feel the *need* for category theory.

I assume that you have some background in algebra and algebraic topology. In several places, I will use concepts from homological algebra, and I recommend Chuck Weibel's book [**W94**] if you need background on that.

How you read this book heavily depends on your background and I therefore refrain from giving a *Leitfaden*. If you know some basic category theory, you might jump ahead to the applications, and if you then realize that for a specific topic you need to look things up, then you can go back to the corresponding spot in the first part of the book. Similarly, some of you might know what a simplicial object is. Then of course you should feel free to skip that section.

I assume that the axiom of choice holds. I will *not* give an introduction to set theory in this book. For diagram categories, I will assume smallness in order to avoid technical problems.

Whenever you see a $A \subset B$ that means $a \in A \Rightarrow a \in B$, so that's what other people might denote by $A \subseteq B$. By $\square$, I denote the end of a proof.

Acknowledgments: This book had a long gestation period. It all started with a lecture course that I gave in Hamburg in the academic term of 2010/11.

I thank Clemens Heine for encouraging me to write the book and Inna Zakharevich who persuaded me to actually finish it. I would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program *Homotopy Harnessing Higher Structures*, supported by EPSRC grant number EP/R014604/1. I finished a first draft when I was in Cambridge in 2018. I was an associate member of the SCR of Churchill College Cambridge, and I am grateful for the hospitality and the welcoming atmosphere in Churchill College. This really helped me to transform a rough draft of the book into a final version.