Introduction

Daniel Gray Quillen gave graduate lectures at the Mathematical Institute in St Giles in Oxford as part of his duties as Waynflete Professor of Pure Mathematics. The audience consisted of postgraduate students, researchers and faculty members, mainly with interests in differential geometry, algebra and functional analysis. At the time, Yang–Mills theory and index theory for operators were pervasive topics in Oxford mathematics. The courses typically lasted for 16 lectures, and were intended as self-contained introductions to the research papers that Quillen was writing. Some of the material fed directly into the papers, whereas other ingredients included motivation and applications which do not feature in the published work. The latter is valuable to a newcomer to cyclic theory, and these notes emphasize the motivation and applications. Quillen kept a mathematical diary which records how he worked over topics to refine essential ideas and developed a profound understanding of diverse theories. The diaries show a remarkable breadth of interest, covering topics in several branches of mathematics. Quillen delivered lectures in traditional style, with calculations written in full detail, so the blackboard was soon filled with tensor products and commuting diagrams. As far as I am aware, he did not produce a collated set of lecture notes, and the record is primarily based upon my notes from the lectures.

The material in Chapters 3 and 4 of this volume is taken from the course ‘Topics in K-theory and cyclic homology’ which Quillen gave in Hilary term 1989, followed by ‘Topics in K-theory and cyclic cohomology’ in Michaelmas 1989. The Hilary 1989 course also covered material which I defer until Chapters 8 and 11, since it required more by way of prerequisites. The title of the present book reflects the emphasis of the courses on cyclic theory and its connection with other branches of mathematics. A special issue of the *Journal of K-Theory* **11** (3) (2013) describes the mathematical legacy of Quillen's work, particularly his contribution to K-theory. Segal [94] describes Quillen's professional career.

The original lectures were not always linearly ordered, as Quillen would revisit calculations and improve them, and digress into topics which were intended to provide motivation for forthcoming material. A significant simplification evidently emerged through his collaboration with Joachim Cuntz, and part of this work is produced here in Chapters 6 and 7, which is taken from the course 'Cyclic cohomology and Karoubi operators' given in Hilary term 1991, and updated in Trinity term 1992. The latter course also covered the material in Chapter 8, regarding connections, and Chapter 12 on Hodge decompositions.

Cyclic theory is an aspect of noncommutative geometry. Classical geometry uses commutative algebra, particularly to describe geometrical objects with additional structure such as differentiability or smoothness. Noncommutative geometry involves noncommutative algebras and seeks to describe geometrical objects which are possibly rough, or topologically complicated. In order to understand the basic definitions of cyclic theory, it is helpful to review some of the definitions of differential geometry that point towards the noncommutative theory.

These notes are intended as an introduction to some topics in cyclic theory and are assuredly not a systematic review. Apart from the historical interest in this material, Quillen's lecture presentations seem to me more accessible than the papers and emphasize the motivation of, and output from, cyclic theory. Some readers might be surprised at the apparent lack of generality, particularly that all the calculations are carried out over fields of characteristic zero, with $\mathbf{C}$ as the default choice of field. Generally, the presentation emphasizes analysis and differential geometry, rather than topology and homological algebra. Quillen motivated some of his results on noncommutative differential forms by comparing the theory with the algebraic approach to Kähler differential forms on algebraic varieties, as we discuss in Chapter 6. This may reflect the interests of Quillen's likely audience for the lectures, or his choice of contemporaneous reading material. His earliest researches were in the formal theory of partial differential equations, and he developed his interest in quantum mechanics and field theory. I have attempted to fill in some of the analytical details omitted from the lectures without making a meal of them, so the reader at least can appreciate what the analytical issues are and how they can be addressed. In so doing, I have used methods that can be readily understood from the viewpoint of a student; I have not introduced more difficult concepts such as metric tensor products, Lie group representations, diffusions on manifolds or Kasparov's KK theory.

There are some differences between Connes's cyclic theory and Quillen's which are worth highlighting. The Chern character is central to Connes's approach, and his B operator has a natural interpretation as a boundary operator. In these notes, the Chern character is mainly discussed for commutative algebras that arise in geometry, and the definition in Chapter 8 is taken from differential geometry, using connections. Connections are fundamental to the development of cyclic theory in this book. In Chapter 7, B is introduced via the Karoubi operator κ, and the properties of B emerge from some simple algebraic computations which do not reveal the geometrical motivation or interpretation. Instead, B and κ are used to turn graded complexes of cochains into modules over principal ideal domains or local rings. Quillen's approach to curvature and commutators fits neatly with the analytical theory of Helton and Howe, as mentioned in Chapters 3 and 9. Some of the results in Chapter 4 regarding algebraic approaches to dilations and extensions appear to be original contributions by Quillen that have the potential for further development. Chapters 6 and 7 emphasize both the similarities and differences between commutative differential forms and noncommutative differential forms. As motivation, we mention results about coordinate rings on algebraic varieties and Riemann surfaces in particular.

Quillen was an assiduous craftsman of mathematics, who refined manuscripts through several stages before publication. The present notes have not undergone such a process and should be read as an outline of his ideas, rather than the finished product. My main purpose in writing these notes is to make the material available to another generation of mathematicians in the hope that they can realize the potential for further development. Some of the topics are of current research interest and there is a legacy of resistant problems which deserve further consideration. For instance, projective modules over Banach algebras are still mysterious. The references do not cover more recent results on cyclic theory or operator K-theory. The first nine chapters should be accessible to any reader with a basic graduate-level knowledge of commutative algebra, differential geometry and functional analysis.

I am especially grateful to Jean Quillen for granting permission on behalf of the Quillen Estate to proceed with publication. Also, I am grateful to Glenys Luke and Roger Astley for encouragement with this project, to James Groves for inspecting the manuscript, and to Andrey Lazarev for helpful comments on an earlier draft. I record my belated but sincere thanks to Merton College Oxford, who supported the junior research fellowship that enabled me to attend Quillen's lectures. More recently, the University of Macau supported the writing of these notes through a Senior Visiting Fellowship.

Gordon Blower
Lancaster University

1
Background Results

The contents of this chapter are classical, in that they refer to the algebraic structures that are used to describe classical mechanics and often involve modules over commutative algebras as in [8]. Some of the results are used in subsequent chapters, while other aspects will be generalized to quantum mechanics and to noncommutative algebras in subsequent chapters. Advanced readers can proceed to Chapter 2.

1.1 Graded Algebras

A unital algebra over a field $\mathbf{k}$ can be regarded as triple $(A, m, 1)$ where $m\colon A \times A \to A$ is the associative multiplication and 1 is the multiplicative unit. The associativity law $(ab)c = a(bc)$ says that the following diagram commutes

$$\begin{array}{ccccccc} A\otimes A\otimes A & \longrightarrow & A\otimes A & \quad & a\otimes b\otimes c & \mapsto & a\otimes bc \\ \downarrow & & \downarrow & & \downarrow & & \downarrow\ ; \\ A\otimes A & \to & A & & ab\otimes c & \to & abc \end{array} \tag{1.1.1}$$

the operation of scalar multiplication on the right gives a commuting diagram

$$\begin{array}{ccccccc} A\otimes A & \longrightarrow & A & \quad & a\otimes \kappa 1 & \longrightarrow & \kappa a \\ \downarrow & & \downarrow & & \downarrow & & \downarrow\ ; \\ A\otimes \mathbf{k} & \to & A & & a\otimes \kappa & \to & a\kappa \end{array} \tag{1.1.2}$$

a similar diagram applies to multiplication from the left. Hence we identify $a \otimes \kappa b$ with $\kappa a \otimes b$, and as a default we work with $A \otimes_{\mathbf{k}} A$, the tensor products over $\mathbf{k}$.

An ideal I of A is a subalgebra such that $am \in I$ and $ma \in I$ for all $a \in A$ and $m \in I$. The square of I is $I^2 = \mathrm{span}_{\mathbf{k}}\{mp\colon m, p \in I\}$, which is also an ideal with $I^2 \subseteq I \subseteq A$, and we can proceed to form $I^3, I^4, \dots$ likewise.

Definition 1.1.1 (i) (*Graded algebra*) Let $\mathbf{k}$ be a field and A a unital algebra over $\mathbf{k}$. Suppose that $(A_n)_{n=0}^{\infty}$ is a sequence of $\mathbf{k}$-vector subspaces of A such that $A = \oplus_{n=0}^{\infty} A_n$, in the sense that each $a \in A$ has a unique expression as a finite sum $a = \sum_{j=0}^{\infty} a_j$ where $a_j \in A_j$ and the multiplication satisfies $A_m A_n \subseteq A_{m+n}$. We suppose that $A_0 = \mathbf{k}1$.

(ii) (*Principal ideal domain*) A commutative ring R with 1 is an integral domain if $xy = 0$ implies $x = 0$ or $y = 0$. Suppose further that every ideal I in R has the form $I = \{rx\colon r \in R\} = (x)$ for some $x \in R$. Then R is called a principal ideal domain (PID).

(iii) An ideal I in a ring R is called nilpotent if $I^n = 0$ for some $n \in \mathbf{N}$.

Example 1.1.2 (Polynomial algebra) (i) Let $\mathbf{k}$ be a field and t an indeterminate. Then the polynomial algebra $\mathbf{k}[t] = \{\sum_{j=0}^{m} a_j t^j : a_j \in \mathbf{k}, m = 0, 1, \dots\}$ with the usual addition and multiplication is a PID. Also with $A_m = \{a_m t^m : a_m \in \mathbf{k}\}$ we obtain a graded algebra. Then $\sum_{j=0}^{m} A_j$ is the space of polynomials of degree less than or equal to m, including the zero polynomial.

(ii) Let R be a unital algebra which is an integral domain, and let $t_1, \dots, t_n$ be commuting indeterminates. Then the algebra of polynomials $A = R[t_1, \dots, t_n]$ is a graded algebra where

$$A_m = \left\{ \sum a_{m_1,\dots,m_n} t_1^{m_1} \dots t_n^{m_n} : a_{m_1,\dots,m_n} \in A, \sum_{j=1}^{n} m_j = m \right\} \tag{1.1.3}$$

and where m is called the total degree, and the elements of A_m are called homogeneous components of degree m.

This R has derivatives ∂_j such that $\partial_j\colon A \to A$ is $\mathbf{k}$-linear and

$$\partial_j a_{m_1,\dots,m_n} t_1^{m_1} \dots t_n^{m_n} = m_j a_{m_1,\dots,m_n} t_1^{m_1} \dots t_j^{m_j - 1} \dots t_n^{m_n}, \tag{1.1.4}$$

so Leibniz's rule holds

$$\partial_j (fg) = (\partial_j f) g + f(\partial_j g) \qquad (f, g \in A) \tag{1.1.5}$$

$\partial_j\colon A_m \subseteq A_{m-1}$ and $\partial_j \partial_k = \partial_k \partial_j$, and ∂_j is an operator of degree (-1). This is the fundamental example of a commutative graded algebra, and may be compared to the following one.

Example 1.1.3 (Formal power series) (i) For a given field $\mathbf{k}$, let $A = \mathbf{k}[[\hbar]]$ be the algebra of formal power series in $\hbar$. Then there is a derivation $\delta\colon \mathbf{k}[[\hbar]] \to \mathbf{k}[[\hbar]]$ given by formal differentiation

$$\delta\colon \sum_{n=0}^{\infty} k_n \hbar^n \mapsto \sum_{n=1}^{\infty} n k_n \hbar^{n-1}. \tag{1.1.6}$$

Let $I = \langle \hbar \rangle$ be the ideal generated by $\hbar$. Then $I^n = \langle \hbar^n \rangle$, and A/I^n is isomorphic to $\mathrm{span}\{1, \hbar, \dots, \hbar^{n-1}\}$ as a $\mathbf{k}$-vector space. Then A/I^n is an Artinian algebra, which has a unique maximal ideal $(\hbar + I^n)$, which is nilpotent, since $(\hbar + I^n)^n = (0)$. Observe that $I^n I^m \subseteq I^{m+n}$ and $A \supset I \supset I^2 \supset \cdots$ is an infinite strictly decreasing sequence of ideals with $\cap_{n=1}^{\infty} I^n = \{0\}$.

There is a derivative $\partial\colon A \to A$ that satisfies $\partial(fg) = (\partial f)g + f\partial g$, and $\partial I^n \subset I^{n-1}$.

(ii) Now let $A[\hbar^{-1}] = \{\sum_{j=-n}^{\infty} a_j \hbar^j; n \in \mathbf{N}; a_j \in \mathbf{k}\}$ be the algebra of formal Laurent series with coefficient in $\mathbf{k}$ that have only finitely many nonzero terms. Then $A_n = \{\sum_{j=-n}^{\infty} a_j \hbar^j; a_j \in \mathbf{k}\}$ is the space of formal Laurent series that have a pole of order at most n at $\hbar = 0$, and A_n is a module over A.

Exercise 1.1.4 (Gradings by powers of an ideal) Let I be an ideal in a unital algebra A over a field $\mathbf{k}$ of characteristic zero. There is a natural filtration by powers of the ideal

$$A \supseteq I \supseteq I^2 \supseteq I^3 \supseteq \cdots$$

called the I-adic filtration. We also impose the condition $\cap_{n=1}^{\infty} I^n = \{0\}$.

(i) Show that there is a natural multiplication map

$$I^n/I^{n+1} \times I^m/I^{m+1} \to I^{n+m}/I^{n+m+1}. \tag{1.1.7}$$

(ii) Deduce that with $I^0 = A$, there is a graded algebra

$$g_I(A) = \bigoplus_{j=0}^{\infty} I^j/I^{j+1}. \tag{1.1.8}$$

(iii) Suppose that A is commutative and I is a maximal ideal. Show that $\mathbf{k} = A/I$ is a field, and $g_I(A)$ is a direct sum of $\mathbf{k}$-vector spaces.

(iv) Suppose that A is commutative and I is a finitely generated maximal ideal. Show that the summands in $g_I(A)$ are finitely generated. (See [8] Theorem 11.22 for more on the structure of $g_I(A)$.)

(v) Show that I is an ideal of R with $I^2 = 0$, where

$$I = \left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} : B \in M_n(\mathbf{C}) \right\} \quad R = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} : A, B, D \in M_n(\mathbf{C}) \right\}.$$

1.2 Derivations

Let A and R be algebras over a field $\mathbf{k}$. A homomorphism is a map $\rho\colon A \to R$ such that

$$\rho(\lambda a + \mu b) = \lambda\rho(a) + \mu\rho(b), \tag{1.2.1}$$

$$\rho(ab) = \rho(a)\rho(b) \qquad (a, b \in A; \lambda, \mu \in \mathbf{k}). \tag{1.2.2}$$

If A and R are unital, and $\rho(1) = 1$, then ρ is said to be unital. Many of the algebras we consider are not unital.

The kernel (nullspace) $\{a\colon \rho(a) = 0\}$ of any homomorphism $\rho\colon A \to R$ is an ideal, and conversely, any ideal arises from the kernel of some homomorphism. Given any ideal I of A, the quotient space $R = A/I$ is an algebra with the usual multiplication and addition, and $\pi\colon A \to R\colon\ a \mapsto a + I$ is a homomorphism with kernel I; we call this the canonical (quotient) homomorphism, and write

$$0 \longrightarrow I \longrightarrow A \longrightarrow R \longrightarrow 0. \tag{1.2.3}$$

Let M be a bimodule over A. This means that M is a $\mathbf{k}$-vector space with operations $A \times M \to M$ and $M \times A \to M$ such that

$$\text{(i)} \quad a(m+n) = am + an; \qquad (m+n)a = ma + na; \tag{1.2.4}$$

$$\text{(ii)} \quad (a+b)m = am + bm; \qquad m(a+b) = ma + bm; \tag{1.2.5}$$

$$\text{(iii)} \quad a(bm) = (ab)m; \qquad (ma) = m(ab); \tag{1.2.6}$$

$$\text{and (iv)} \quad a(mb) = (am)b \qquad (a, b \in A; m, n \in M). \tag{1.2.7}$$

If A has a unit 1 we generally suppose that $1m = m1 = m$ for all $m \in M$.

In particular, A is a bimodule over A with the left and multiplication operations $(a, b) \mapsto ab$. Likewise, any ideal I of A is an A-bimodule. Given A-modules M and N, we write $\mathrm{Hom}_A(M, N)$ for the space of left A-module maps $\psi\colon M \to N$ such that $\psi(am + np) = a\psi(m) + b\psi(p)$ for all $a, b \in A$ and $m, p \in M$.

Definition 1.2.1 (Derivation) A derivation is a map $\delta\colon A \to M$ such that

$$\delta(\lambda a + \mu b) = \lambda\delta(a) + \mu\delta(b), \tag{1.2.8}$$

$$\delta(ab) = a\delta(b) + \delta(a)b \qquad (a, b \in A; \lambda, \mu \in \mathbf{k}). \tag{1.2.9}$$

We write the space of such derivations as $\mathrm{Der}_{\mathbf{k}}(A, M)$ to emphasize that $\mathbf{k}$ is a field of constants. Given any bimodule M over A, we can choose $m \in M$ and introduce the inner derivation $\delta_m\colon A \to M$ by

$$\delta_m(a) = [a, m] = am - ma. \tag{1.2.10}$$

It is of interest to determine the derivations that can be expressed in this form; see Section 2.6.

Example 1.2.2 (Derivations on polynomials) Let $\mathbf{k}[t]$ be the polynomial algebra over $\mathbf{k}$ with the derivation $\partial\colon \mathbf{k}[t] \to \mathbf{k}[t]$ given by the usual derivative $\partial(\sum_{j=0}^n a_j t^j) = \sum_{j=1}^n j a_j t^{j-1}$. Next let dt be the formal infinitesimal with $(dt)^2 = 0$, and introduce $\mathbf{k}[t, dt] = \mathbf{k}[t] \oplus \mathbf{k}[t]dt$ with the multiplication

$$\big(f_0(t)+ f_1(t)dt\big)\big(g_0(t) + g_1(t)dt\big) = f_0(t)g_0(t) + (f_0(t)g_1(t) + f_1(t)g_0(t))dt,$$

so that $R = \mathbf{k}[t, dt]$ is a commutative algebra with ideal $I = \mathbf{k}[t]dt$ which has square zero in the sense that $I^2 = 0$, and we can identify $A = \mathbf{k}[t]$ with R/I. Also, let $d(f_0(t) + f_1(t)dt) = \partial f_0(t)dt$. Then R is an A-bimodule, and $d\colon A \to R$ gives a derivation.

Example 1.2.3 (Pseudo-differential operators) (i) Let A be a unital commutative ring, and let $\partial\colon A \to A$ be a derivation. Then the space of formal differential operators with coefficients in A is given by

$$DO_A = \left\{ \sum_{j=0}^n a_j \partial^j : n \in \mathbf{N}, a_j \in A \right\}$$

which forms a ring under the composition rule

$$\left(\sum_{j=0}^n a_j \partial^j\right) \circ \left(\sum_{k=0}^m b_j \partial^k\right) = \sum_{j,k,\ell:0\le \ell\le j\le n, 0\le k\le m} \binom{j}{k} a_j \big(\partial^\ell b_k\big) \partial^{j+k-\ell}$$

so that A is a subring of DO_A. Now we introduce the integration operator ∂^{-1}, which is required to satisfy

$$\partial^{-1} \circ a = a\partial^{-1} - (\partial a)\partial^{-2} + (\partial^2 a)\partial^{-3} - \cdots,$$

and $\partial \circ \partial^{-1} = 1$, so $\partial \circ \partial^{-1} \circ a = a$ for all $a \in A$. As in [70], one can extend the composition rule to powers ∂^j with $j \in \mathbf{Z}$ via the generalized Leibniz rule. We introduce

$$\Psi DO_A^n = \left\{ \sum_{j=-\infty}^n a_j \partial^j : a_j \in A \right\} \qquad (n \in \mathbf{Z}).$$

Then $\Psi DO_A = DO_A \oplus \Psi DO_A^{-1}$ gives a graded algebra such that $\Psi DO_A^n \subseteq \Psi DO_A^{n+1}$ with $\partial \circ \Psi DO_A^j \subseteq \Psi DO_A^{j+1}$ and

$$\Psi DO_A^j \circ \Psi DO_A^k \subseteq \Psi DO_A^{j+k} \qquad (j, k \in \mathbf{Z}).$$

For later use, we note that the commutator $[X,Y] = X \circ Y - Y \circ X$ has the special property

$$[\Psi DO_A^j, \Psi DO_A^k] \subset \Psi DO_A^{j+k-1} \qquad (j,k \in \mathbf{Z}),$$

so we lose a derivative when taking the commutator. There is an exact sequence of algebras

$$0 \longrightarrow \Psi DO_A^{-1} \longrightarrow \Psi DO_A^0 \longrightarrow A \longrightarrow 0.$$

Mulase [76] considered this as a graded algebra.

(ii) (*Weyl algebra*) With $A = \mathbf{C}[z]$, the algebra of differential operators with polynomial coefficients $DO_A = \mathbf{C}\langle z, d/dz\rangle$ is known as the Weyl algebra; in Definition 9.2.1 we obtain this algebra via a more sophisticated construction. As in (i), we can extend DO_A to $\Psi DO_A = \mathbf{C}\langle z, d/dz, (d/dz)^{-1}\rangle$.

1.3 Commutators and Traces

Definition 1.3.1 (i) (*Commutators*) Let A be an associative algebra over $\mathbf{k}$, and M an A-bimodule. Then we write $[a,m] = am - ma$ for the commutator of $a \in A$ and $m \in M$, then let

$$[A,M] = \mathrm{span}_{\mathbf{k}}\{[a,m]; \quad a \in A, m \in M\}$$

for the commutator subspace, which is a $\mathbf{k}$-vector subspace of M. The commutator quotient space is $M/[A,M]$.

(ii) (*Traces*) In particular, A is an A-bimodule for the standard multiplication, so $[A,A]$ is the subspace of A spanned by the commutators. Then a trace $\tau : A \to \mathbf{k}$ is a $\mathbf{k}$-linear map such that $\tau([a,b]) = 0$ for all $a,b \in A$ equivalently, a trace is a linear function $\tau : A/[A,A] \to \mathbf{k}$.

Let I be an ideal in an associative algebra A, so that I is an A-bimodule. Then $[I,I]$ and $[I,A]$ are commutator subspaces of I. A trace on I is a linear functional $\tau : I \to \mathbf{k}$ such that $\tau|[I,I] = 0$. Such a trace may have the stronger property that $\tau|[I,M] = 0$. See Proposition 3.2.9(iv) for a significant example.

Example 1.3.2 (Quaternions) (i) We consider an example of a noncommutative algebra, namely the quaternions. We introduce Pauli's matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{1.3.1}$$

which satisfy

$$\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I; \qquad \sigma_j\sigma_k = -i\sigma_\ell \tag{1.3.2}$$

for all cyclic permutations $(jk\ell)$ of (123). Note that $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, i\sigma_0, i\sigma_1, i\sigma_2, i\sigma_3\}$ gives the quaternion group of eight elements. The space $\mathbf{H} = \{a_0\sigma_0 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 : a_j \in \mathbf{R}\}$ gives the algebra of quaternions, a four-dimensional noncommutative division ring over $\mathbf{R}$. The elements with $a_0 = 0$ are called pure quaternions.

(ii) Let $\mathbf{R}^3$ have the standard unit vector basis $\{e_1, e_2, e_3\}$ and let $\mathbf{H} \to \mathbf{R} \times \mathbf{R}^3$ be the linear map $a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 \to (a_0, \mathbf{a})$ where $\mathbf{a} = (a_1, a_2, a_3)$. We write $(a_0, \mathbf{a})^* = (a_0, -\mathbf{a})$, and introduce the multiplication

$$(a_0, \mathbf{a})(b_0, \mathbf{b}) = (a_0b_0 - \mathbf{a}\cdot\mathbf{b}, a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a}\times\mathbf{b}) \tag{1.3.3}$$

where $\mathbf{a}\cdot\mathbf{b}$ is the usual scalar (dot) product of vectors and $\mathbf{a}\times\mathbf{b}$ is the usual vector (cross) product on $\mathbf{R}^3$. This $*$ gives an anti-automorphism of the real skew ring $\mathbf{H}$, such that

$$(q_1 + q_2)^* = q_1^* + q_2^*; \qquad (q_1q_2)^* = q_2^*q_1^*.$$

Then we define a norm by $\|(a_0, \mathbf{a})\| = \sqrt{a_0^2 + \mathbf{a}\cdot\mathbf{a}}$. By vector algebra, one checks that

$$\|(a_0, \mathbf{a})(b_0, \mathbf{b})\| = \|(a_0, \mathbf{a})\|\|(b_0, \mathbf{b})\|. \tag{1.3.4}$$

If $(a_0, \mathbf{a}) \in \mathbf{H}$ has $\|(a_0, \mathbf{a})\| = 1$, then we say that $(a_0, \mathbf{a})$ is a unit quaternion. One checks that $Sp(1) = \{(a_0, \mathbf{a}) \in \mathbf{H} : \|(a_0, \mathbf{a})\| = 1\}$ is a group.

Now observe that

$$a_0\sigma_0 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{bmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{bmatrix}$$

so that there is a bijective group homomorphism map between the group $Sp(1)$ of unit quaternions and the group $SU(2)$ of 2×2 unitary complex matrices that have determinant one; $Sp(1) \sim SU(2, \mathbf{C})$.

(iii) (*Rotations and spin*) Let $SO(3)$ be the group of all rotations of Euclidean space about the origin, namely $\{U \in M_3(\mathbf{R}) : U^tU = I, \det U = 1\}$ with matrix multiplication. There is an exact sequence of groups

$$0 \to \mathbf{Z}/(2) \to Sp(1) \to SO(3) \to 0, \tag{1.3.5}$$

arising as follows. For all $p \in Sp(1)$, there exists a real algebra automorphism of $\mathbf{H}$ given by conjugation $\alpha_p(q) = pqp^{-1}$. Note that $\alpha_p \circ \alpha_r = \alpha_{pr}$, so $p \mapsto \alpha_p$ is a group action; evidently the kernel is $\{\pm 1\}$. Conversely, given any real automorphism α of $\mathbf{H}$, α acts on the pure quaternions so that