

Thinking Probabilistically

Probability theory has diverse applications in a plethora of fields, including physics, engineering, computer science, chemistry, biology, and economics. This book will familiarize students with various applications of probability theory, stochastic modeling, and random processes, using examples from all these disciplines and more.

The reader learns via case studies and begins to recognize the sort of problems that are best tackled probabilistically. The emphasis is on conceptual understanding, the development of intuition, and gaining insight, keeping technicalities to a minimum. Nevertheless, a glimpse into the depth of the topics is provided, preparing students for more specialized texts while assuming only an undergraduate-level background in mathematics. The wide range of areas covered – never before discussed together in a unified fashion – includes Markov processes and random walks, Langevin and Fokker–Planck equations, noise, generalized central limit theorem and extreme values statistics, random matrix theory, and percolation theory.

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Stochastic Processes, Disordered Systems,
and Their Applications

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