# 1 Space and Time

## Orientation and Euler Angles

**Kinematics** deals with the *geometry of motion of material bodies along time* – change of position in space and over time – without regard to the physical phenomena on which it depends. Such description requires mathematical models for space and time (which is the physical framework of mechanical phenomena) and mathematical models for bodies.

There is a range of models of increasing complexity for material objects: mass point (or particle), rigid body, continuous media (including deformable bodies and fluids). This text deals exclusively with the first two (Chapters 2 and 3), whose mathematical complexity is much lower than that required by continuous media.

The mathematical operation *time derivative*, to evaluate rates of change (of position, orientation, etc.), is a fundamental tool in mechanics. The time derivative of vectors is more complex than that of scalars, since it has to take into account both the change of value and of orientation. While the former is identical for all observers (or all *reference frames*), the latter is not. For example, a radius marked on a rotating platform (and taken as a vector) has the same value and orientation for observers fixed to the platform but variable orientation for observers fixed to the ground.

Orientation is a key concept, and its mathematical description is not trivial when 3D motion is addressed. In this text, only the Euler angles are used as they are the most appropriate procedure for a first study. However, Appendix 1B presents a brief review of some alternative rotation parameters due to its importance in engineering branches such as robotics and spacecraft dynamics.

Operations on vectors can be done through their graphic representation (an arrow with value indication[1]) or their components (projections on a vector basis). In mechanics, *vector bases of variable orientation relative to the reference frame* (*moving bases*) are usual because they facilitate the vectors' projection and simplify the expressions of their components and their physical interpretation. For instance, when studying the kinematics of a vehicle, the basis with constant orientation relative to the chassis is particularly interesting.

The time derivative of vectors expressed through their components in a moving basis leads to the concept of *angular velocity* of the vector basis, which describes its rate of change of orientation relative to the reference frame. The angular velocity is a key concept in rigid body kinematics (Chapter 3).

[1] *Value* and *module* (or *magnitude*) of a vector are not synonyms: while the former may be positive or negative, the latter is strictly positive.

## 1.1 The Absolute Time of Newtonian Mechanics

In mechanics, the concept of time is linked to that of *ordered succession of instants*. An *instant t* is what two simultaneous events have in common. Let us suppose that an observer detects two events (for example, the collision of particles $P_1$ and $P_2$ – event A – and the collision of particles $P_3$ and $P_4$ – event B). The observer can establish the following without ambiguity:

- If they are simultaneous, $t_A = t_B$
- If A precedes B, $t_A < t_B$
- If B precedes A, $t_A > t_B$

From the ordering of events, we can establish an ordered sequence of instants that we call *time*.

The **principle of absolute simultaneity** states that this sequence is the same for all observers whose relative speed is much lower than that of light (which is the case considered in Newtonian mechanics): Newtonian time is an **absolute time**. A same clock ticks the time instants for all observers – for example, by showing the succession of coincidences of a needle with marks on a dial. Since it is an ordered and dense succession of points (between two instants it is always possible to insert another), the mathematical model for time is the one-dimensional space of real numbers $R^1$.

The definition of time must be completed with its measurement (that is, the assessment of the time interval between two instants). This is not just a kinematic issue but a dynamic one (**principle of inertia**, in section 1.3, chapter 1 of *Rigid Body Dynamics* [Cambridge University Press, forthcoming]).

In relativistic mechanics, which holds when the relative speed between observers is close to that of light, the principle of absolute simultaneity is no longer valid: two events A and B can be simultaneous for an observer but be "A before B" for a second observer and "B before A" for a third observer. For this reason, each observer needs its own clock.

## 1.2 Space and Reference Frame

In Newtonian mechanics, the physical space is modeled as an affine Euclidean three-dimensional point space $E^3$. To define the location of points of $E^3$, we need to choose a point **O** (called **origin**) and three noncoplanar axes (Fig. 1.1). Vectors $\overline{\mathbf{OP}}$ (where **P** is any point in $E^3$) are position vectors, and they belong to the vector space associated with the affine point space.

An origin **O** and a set of three noncoplanar axes define a **mathematical frame of reference** in $E^3$. A different origin or a different set of axes yields a different description of those position vectors and hence defines a different frame of reference from the mathematical point of view.[2]

[2] The mathematical concept of reference frame (MRF) and that of coordinate system are not equivalent: for the same MRF, different coordinate systems (Cartesian, polar, cylindrical, etc.) can be used.

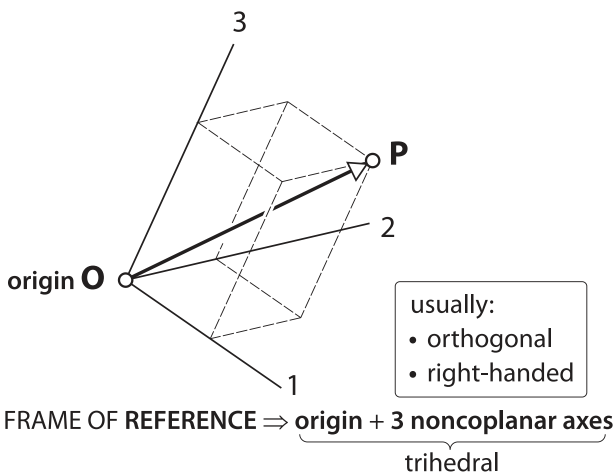


**Fig. 1.1**

In mechanics, the definition of frame of reference is different: it is a set of "rest points" (points whose mutual distances do not change – they are **mutually fixed**) in $E^3$. Note that this definition contains implicitly the idea of **state of motion** and thus the concept of time (not taken into account in the mathematical reference frame). The constitutive elements of physical reference frames are space and time.

This physical notion is intrinsically linked to that of the **observer**. The observer (which is not a point) may be located anywhere, and what he/she measures is the movement of points relative to the reference frame.

Movement is always associated with a reference frame: movements are **relative** to reference frames. If a point **P** passes through different points of a reference frame R, it is a moving point relative to R. The set of those points constitutes the **trajectory of P in R**. In principle, a same point describes different trajectories in different reference frames.

A reference frame R can be graphically represented either through a set of points mutually fixed or through a **trihedral** (single rest point **O** in R and three noncoplanar axes, Fig. 1.2). Sometimes, the observer is added to the representation of the reference frame (Fig. 1.3). Note that no clock is included in those compact representations. This is so because we will be dealing with Newtonian mechanics, and in that context the *principle of absolute simultaneity* holds. In other words, a same clock is shared by all observers, and so it can be suppressed from the representation without generating any confusion.

Reference frame (R) and **rigid body** are close concepts: both are sets of mutually fixed points, though those in the rigid body are just a subset in $E^3$, and they are **particles** (or mass points) (Fig. 1.4). This is why we usually describe reference frames through rigid body names: rotating platform frame, chassis frame, etc.

A consequence of that equivalence is that the concept of orientation (and that of angular velocity) applies both to reference frames and rigid bodies.

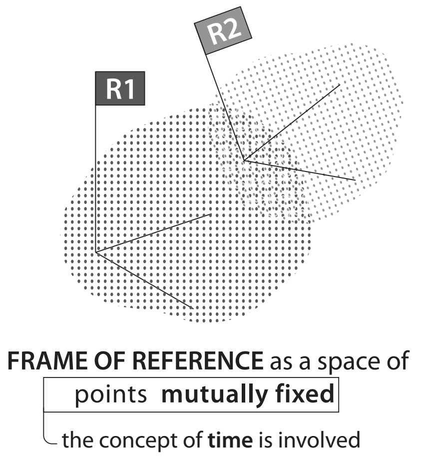


**Fig. 1.2**

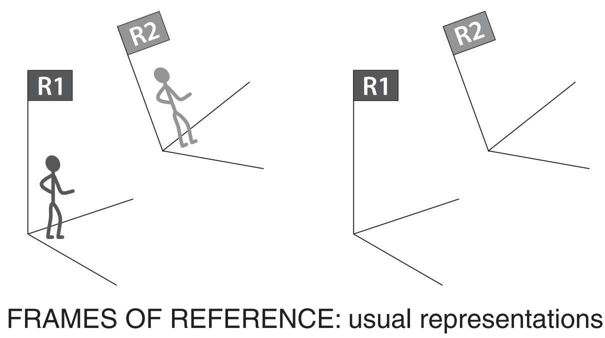


**Fig. 1.3**

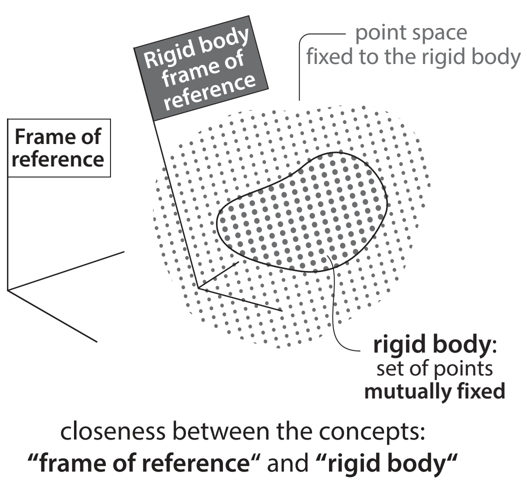


**Fig. 1.4**

## 1.3 Representation of Vectors and Operations Involving a Single Time Instant

In a vector space of three dimensions (as $E^3$), vectors can be represented in a simple and intuitive graphical way as an **arrow** (which defines a **positive generic direction**) and a **value** (defined by a real number). When the value is positive, the vector direction corresponds to that of the arrow; when it is negative, it is the opposite.

The description (arrow, value) is more efficient than that of (**orientation, sense, magnitude**): as the value incorporates magnitude and sense, there is no need of two analytical formulations (one for positive sense and another one for negative).

Graphic representations played an important role before the advent of computers: **graphic statics** and **graphic kinematics** constituted two important branches of mechanics applied to engineering. Easy access to computational power has led to a partial disregard of graphic techniques in favor of numerical treatments.

The graphic representation is widely used in this book as an auxiliary element. In very simple cases, it allows operations (sum, vector product, derivation, etc.) without resorting to analytical treatments based on the representation of the vector through its components in a vector basis.

Vector operations carried out in the same time instant – algebraic operations such as addition, scalar product, and vector product – have a simple geometric description and can be performed from their graphical representation (Fig. 1.5).

The time derivative of a vector, which is an operation over time – it involves the vector in two time instants separated by a time interval that tends to zero – can also be treated directly through its graphic representation (Section 1.4).

All previous operations can be performed through the description of the vectors by their components in a vector basis. The basis orientation can be either constant (**fixed basis**) or variable in the reference frame (**moving basis**). **Whether a basis is fixed or moving is irrelevant in operations** that involve vectors at a same time instant.

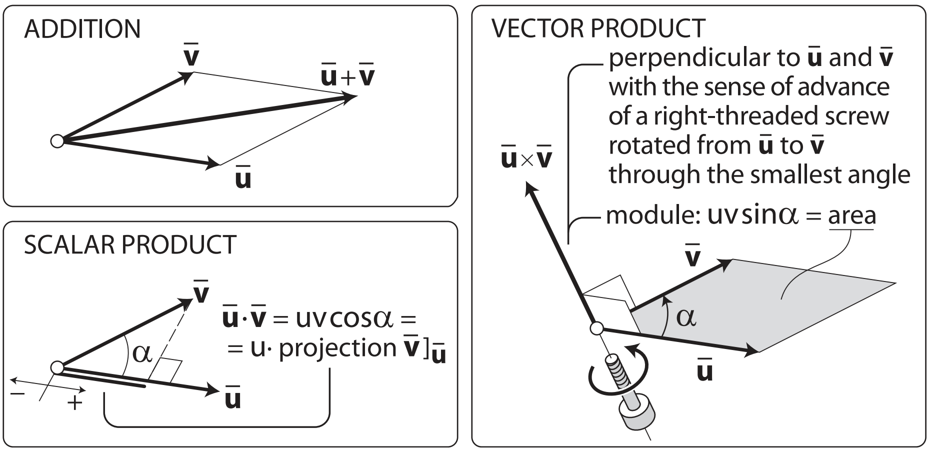


**Fig. 1.5**

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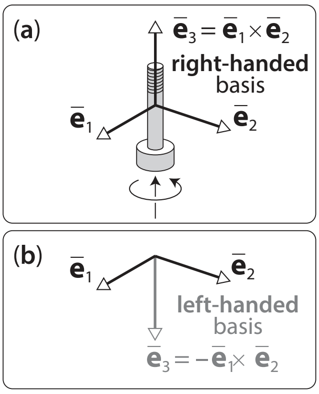


**Fig. 1.6**

In a vector basis B with versors $(\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3)$, a vector $\bar{\mathbf{u}}$ is represented by the column vector of its components $(u_1, u_2, u_3)$:

$$\bar{\mathbf{u}} = \sum_{i=1}^{3} u_i \bar{\mathbf{e}}_i \rightarrow \{\bar{\mathbf{u}}\}_B = \begin{Bmatrix} u_1 \\ u_2 \\ u_2 \end{Bmatrix}. \tag{1.1}$$

In this text, all the bases are:

- Orthonormal: $\bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$
- Right-handed: $\bar{\mathbf{e}}_3 = \bar{\mathbf{e}}_1 \times \bar{\mathbf{e}}_2$ (Fig. 1.6a); if $\bar{\mathbf{e}}_3$ had the opposite direction, it would be an inverse (or left-handed) basis (Fig. 1.6b)

From that analytical representation, the addition and product operations are solved simply as:

$$\{\bar{\mathbf{u}}+\bar{\mathbf{v}}\}_B = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{Bmatrix}, \{\bar{\mathbf{u}}\cdot\bar{\mathbf{v}}\}_B = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \cdot \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{1.2}$$

The cross product can be calculated from a determinant:

$$\bar{\mathbf{u}} \times \bar{\mathbf{v}} = \mathrm{DET} \begin{vmatrix} \bar{\mathbf{e}}_1 & \bar{\mathbf{e}}_2 & \bar{\mathbf{e}}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \Rightarrow \{\bar{\mathbf{u}} \times \bar{\mathbf{v}}\}_B = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \times \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{Bmatrix}. \tag{1.3}$$

The $\bar{\mathbf{u}} \times \bar{\mathbf{v}}$ product can be understood as a linear application $(\bar{\mathbf{u}}\times)$ transforming vector $\bar{\mathbf{v}}$ into vector $\bar{\mathbf{u}} \times \bar{\mathbf{v}}$. Therefore, it can be represented by a matrix (in this case, antisymmetric):[3]

[3] This matrix representation has a practical interest in certain analytical processes (Section 1.5).

$$\{\bar{\mathbf{u}} \times \bar{\mathbf{v}}\}_B = [\bar{\mathbf{u}}\times]_B\{\bar{\mathbf{v}}\}_B = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}. \tag{1.4}$$

The expressions in Eqs. (1.2, 1.3) are valid for any basis B (fixed or moving). This is not the case in operations involving vectors at different time instants (such as time derivatives and integrations). As will be seen in Section 1.6, the analytical time derivative (from the vectors' projection on moving bases) introduces a term that depends on the rate of change of orientation of the basis in the reference frame. Time integration through moving bases is presented in Appendix 1C.

The criterion for choosing a vector basis is of a practical nature: it is convenient that the obtaining of the vectors components is simple, and that the result is simple enough (if possible) to facilitate its physical interpretation.

## 1.4 Geometric Time Derivative of Vectors

The time derivative of a variable (•) evaluates its change per time unit:

$$\frac{d(\bullet)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta(\bullet)}{\Delta t}. \tag{1.5}$$

If the variable is a scalar function ρ (for instance, the distance ρ between two points), the result is identical in all reference frames, as all observers detect the same Δρ for a same Δt (because of the principle of absolute simultaneity):

$$\frac{d\rho}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\rho}{\Delta t}, \text{ same unique value for all observers.} \tag{1.6}$$

However, the change of a vector $\Delta\bar{\mathbf{u}}$ is not the same for all observers in principle. The change of its value is indeed the same for everyone (since it is a scalar), but there may be discrepancies when it comes to evaluating the change of its orientation. For example, if we consider the earth frame RA and the platform frame RB (rotating about an axis fixed in RA, Fig. 1.7), a vector $\bar{\mathbf{u}}$ constant for observer B (with zero time derivative for observer B) is a rotary vector for observer A as the $\bar{\mathbf{u}}$ orientation is not constant. The origin of this discrepancy is the relative rotation between the two reference frames.

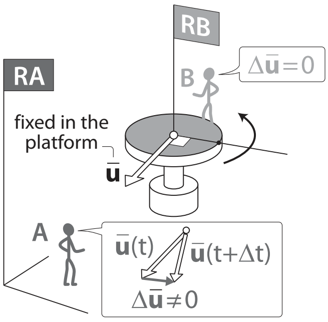


Fig. 1.7

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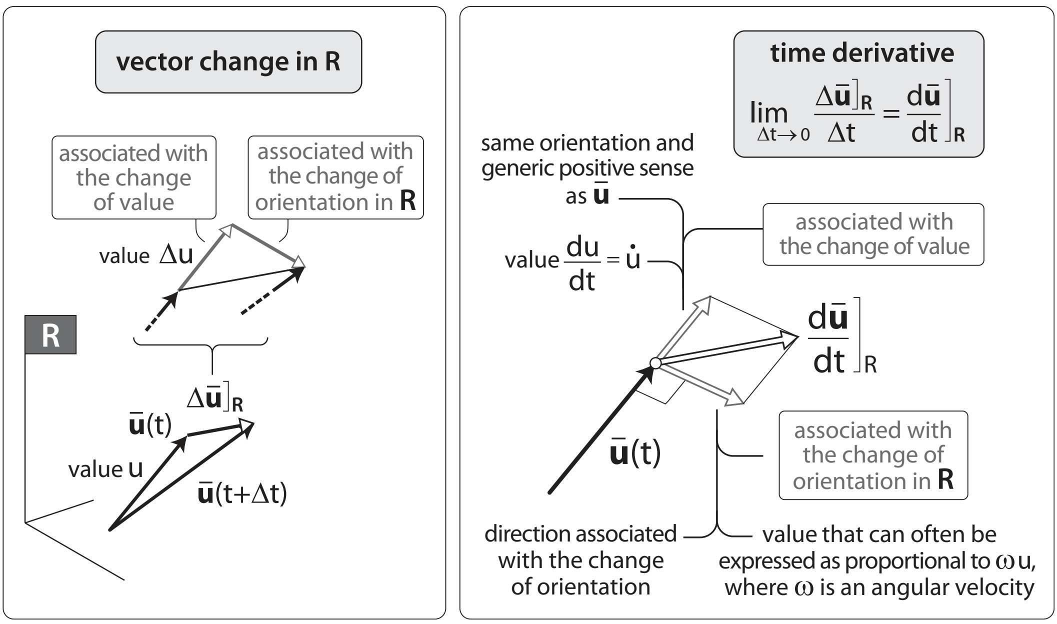


**Fig. 1.8**

The time derivative of a vector, then, is reference-dependent. It is advisable to explicitly include this dependency in the notation:

$$\left.\frac{d\bar{\mathbf{u}}}{dt}\right]_R = \lim_{\Delta t\to 0}\frac{\Delta\bar{\mathbf{u}}]_R}{\Delta t} = \lim_{\Delta t\to 0}\left.\frac{\Delta\bar{\mathbf{u}}}{\Delta t}\right]_R, \tag{1.7}$$

where subscript R is the reference frame where it is calculated. A frequent notation for the time derivative is a simple dot on the variable:

$$\dot{\rho} \equiv \frac{d\rho}{dt} \quad ; \quad \dot{\bar{\mathbf{u}}}]_R \equiv \left.\frac{d\bar{\mathbf{u}}}{dt}\right]_R . \tag{1.8}$$

When there is no doubt about the reference frame where the time derivative is being calculated, the subscript R will be disregarded, and we will just write $\dot{\bar{\mathbf{u}}}$ to simplify the notation.

The time derivative of a vector in a given reference frame R is nonzero when there is a change in value (equal in all R) or a change in orientation (which is R-dependent in principle, Fig. 1.8). When $\Delta t \to 0$, those changes can be associated with two different vectors (whose addition yields the total time derivative):

- The vector associated with the change in value: it is parallel to $\bar{\mathbf{u}}$ and with the same positive generic sense; its value is $\dot{u} = du/dt$
- The vector associated with the change of orientation: it is perpendicular to $\bar{\mathbf{u}}$, with direction associated with the change of orientation; its value can often be expressed as a product of value u (or of some projection of $\bar{\mathbf{u}}$) and an angular velocity ω

This procedure will be called **geometric time derivative** (to distinguish it from the analytic time derivative, based on the components of the vector in a vector basis). It is highly recommended in simple cases not just because it is straightforward but because it is a description close to the physical meaning of the time derivative, so it may help to avoid mistakes that slip easily in the analytical derivative. Chapter 2 presents some examples of this procedure.

## 1.5 Analytical Time Derivative of Vectors: Angular Velocity Vector

The geometric time derivative described in the previous section is a powerful tool, but it may be difficult to use when vectors evolve in a 3D space. In those cases, it is advisable to shift to an analytical calculation (through a vector basis). However, this can be tricky, mainly for two reasons:

- The time derivative calls for the information of the vector in *two different time instants* (whose separation tends to zero); in the analytical calculation, this may go unnoticed (while it never does in the geometric one)
- If the vector is projected in a moving basis MB (relative to a reference frame R), the change of the value of its components does not describe the change of the vector in R as the latter may also be influenced by the change of orientation of MB relative to R

In a vector basis FB with constant orientation (fixed basis) in R, the components of vector $\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}}$ are just the time derivative of the components of $\bar{\mathbf{u}}$:

$$\{\bar{\mathbf{u}}\}_{\mathrm{FB}} = \begin{Bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \\ \mathrm{u}_3 \end{Bmatrix} \rightarrow \left\{\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}}\right\}_{\mathrm{FB}} = \begin{Bmatrix} \dot{\mathrm{u}}_1 \\ \dot{\mathrm{u}}_2 \\ \dot{\mathrm{u}}_3 \end{Bmatrix} \equiv \frac{\mathrm{d}}{\mathrm{dt}}\{\bar{\mathbf{u}}\}_{\mathrm{FB}}. \tag{1.9}$$

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♣ *Proof*

In a fixed basis with versors $(\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3)$, vector $\bar{\mathbf{u}}$ is:

$$\bar{\mathbf{u}} = \sum_{i=1}^{3} \mathrm{u}_i \bar{\mathbf{e}}_i. \tag{1.10}$$

Its time derivative in a reference frame R is:

$$\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}} = \sum_{i=1}^{3} \dot{\mathrm{u}}_i \bar{\mathbf{e}}_i + \sum_{i=1}^{3} \mathrm{u}_i \dot{\bar{\mathbf{e}}}_i\big]_{\mathrm{R}}. \tag{1.11}$$

As the components $\mathrm{u}_i$ are scalar functions, there is no need of explicit indication of the reference frame when calculating $\dot{\mathrm{u}}_i$. This explicit indication is mandatory when performing the time derivative of versors $\bar{\mathbf{e}}_i$. However, as FB is a fixed basis, the time derivative of $\bar{\mathbf{e}}_i$ is zero, so:

$$\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}} = \sum_{i=1}^{3} \dot{u}_i \bar{\mathbf{e}}_i. \tag{1.12}$$

Note that this calculation requires the knowledge of the components along time $(u_i(t))$, or, equivalently, their expression has to be completely general (that is, valid for any time instant). ♣

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Let's consider now the time derivative in R of a vector $\bar{\mathbf{u}}$ projected in a moving basis MB. In that case, a nonzero value of $\dot{u}_i$ does not imply that $\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}}$ is nonzero. In general, that time derivative contains two terms:

$$\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}} = \sum_{i=1}^{3} \dot{u}_i \bar{\mathbf{e}}_i + \sum_{i=1}^{3} u_i \dot{\bar{\mathbf{e}}}_i\big]_{\mathrm{R}}, \quad \dot{\bar{\mathbf{e}}}_i\big]_{\mathrm{R}} \neq 0. \tag{1.13}$$

Equation (1.13) proves that a vector with constant $u_i (\dot{u}_i = 0)$ is not necessarily a vector constant in R:

$$\dot{u}_i = 0 \Rightarrow \dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}} = \sum_{i=1}^{3} u_i \dot{\bar{\mathbf{e}}}_i\big]_{\mathrm{R}} \neq 0. \tag{1.14}$$

If the vector is constant in R:

$$\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}} = 0 \Rightarrow \sum_{i=1}^{3} \dot{u}_i \bar{\mathbf{e}}_i = -\sum_{i=1}^{3} u_i \dot{\bar{\mathbf{e}}}_i\big]_{\mathrm{R}}. \tag{1.15}$$

Equation (1.15) shows that now the only cause of variation of the vector components $(\dot{u}_i)$ is the change of orientation of the MB relative to R $\left(\dot{\bar{\mathbf{e}}}_i\big]_{\mathrm{R}}\right)$.

**When using moving bases, the time derivatives of the components are only a part of the vector time derivative.**

The components of $\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}}$ are obtained by adding the column vector of the time derivatives of the vector components and the vector product $\bar{\mathbf{\Omega}}_{\mathrm{R}}^{\mathrm{MB}} \times \bar{\mathbf{u}}$, where $\bar{\mathbf{\Omega}}_{\mathrm{R}}^{\mathrm{MB}}$ is the **angular velocity of MB relative to R**:

$$\left\{\dot{\bar{\mathbf{u}}}\big]_{\mathrm{R}}\right\}_{\mathrm{MB}} = \frac{\mathrm{d}}{\mathrm{dt}}\{\bar{\mathbf{u}}\}_{\mathrm{MB}} + \left\{\bar{\mathbf{\Omega}}_{\mathrm{R}}^{\mathrm{MB}} \times \bar{\mathbf{u}}\right\}_{\mathrm{MB}}. \tag{1.16}$$

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♣ *Proof*

The vector components of vector $\bar{\mathbf{u}}$ in a fixed basis FB and in a moving basis MB are related through the transformation matrix [S]:

$$\{\bar{\mathbf{u}}\}_{\mathrm{FB}} = [\mathrm{S}]\{\bar{\mathbf{u}}\}_{\mathrm{MB}}. \tag{1.17}$$

The columns in [S] are the components of the MB basis projected in the FB basis. This matrix is variable in time. As the bases are orthonormal, it follows that:

$$[\mathrm{S}]^{-1} = [\mathrm{S}]^{\mathrm{T}}. \tag{1.18}$$