NUMERICAL RANGES OF HILBERT SPACE OPERATORS

Starting with elementary operator theory and matrix analysis, this book introduces the basic properties of the numerical range and gradually builds up the whole numerical range theory. More than 400 assorted problems, ranging from routine exercises to published research results, give you the chance to put the theory into practice and test your understanding. Interspersed throughout the text are numerous comments and references, allowing you to discover related developments and to pursue areas of interest in the literature. Also included is an appendix on basic convexity properties on the Euclidean space. Targeted at graduate students as well as researchers interested in functional analysis, this book provides a comprehensive coverage of classic and recent works on the numerical range theory. It serves as an accessible entry point into this lively and exciting research area.

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Numerical Ranges of Hilbert Space Operators

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To the numerical range community
and
to our beloved wives
Yenlin and Wenyu

Contents

Preface

The study of numerical range starts with the discovery by Toeplitz and Hausdorff 100 years ago (in 1918 and 1919) of the convexity of the numerical ranges (called fields of values in the early years) of finite matrices. Progressing at a slow pace in the beginning, the theory picks up steam gradually over the subsequent years and has now established itself as a full-fledged subfield of operator theory. It provides a powerful tool in studying properties of the operators on a Hilbert space. The converse line of research on the shape of the numerical ranges of operators in various special classes is also very fruitful. Useful references in the past, from which we ourselves learned the theory over the years, are [290, Chapter 22] by Halmos, [314, Chapter 1] by Horn and Johnson, the two monographs [66, 67] by Bonsall and Duncan, and the book [281] by Gustafson and Rao.

The present book is intended to be a comprehensive one covering different aspects of the theoretical developments of the numerical range theory. It originates from a draft prepared by the first author more than two decades ago. Lying idly for all these years, it was resurrected only in the past two years with the joining of the second author. It goes without saying that a great amount of updating, rewriting, plus the adding of Chapters 7 and 8 was needed to incorporate the many newly discovered results of this area into the final edition. At the research level, the book serves as a reference source for various topics in the study of numerical range. Selected parts of the contents are also suitable for a one- or two-term graduate course or a graduate seminar after one on functional analysis or operator theory.

The prerequisite for this book is some basic understanding of the elementary functional analysis or operator theory. A brief summary of the needed results together with the commonly used notations and terms has been given in the Introduction.

The main text starts with Chapter 1 on the basic properties of the numerical range. These include its relations to the spectrum of the concerned operator, the topological properties of its boundary and interior, how it changes under various transforms of the operator, and the different parameters such as the numerical radius, Crawford numbers, and others which measure its size in metric terms.

We then turn our attention, in Chapter 2, to the numerical ranges of certain special class operators such as the quadratic operators, normal operators, hyponormal and

Toeplitz operators, (unilateral or bilateral) weighted shifts, and composition operators on H^2. The chapter ends with the attainment problem, namely, the problem of determining which nonempty bounded convex subset of the plane is the numerical range of some operator on a (separable) Hilbert space. One major result in this respect is the one proved by Agler that numerical ranges must be Borel sets.

Chapter 3 starts with a more detailed study of the numerical radius. The numerical contractions (operators with numerical radii at most one) are then taken up in subsequent sections. The most interesting property in this part is the power inequality that powers of numerical contractions are themselves numerical contractions. The important theorem of Ando's gives a structural factorization of numerical contractions, which enables us to generalize the power inequality from powers to analytic functions of numerical contractions and to improve over Crabb's results concerning the asymptotic behavior of the norms of the powers of a numerical contraction acting on a unit vector. The chapter ends with a discussion on the numerical radius of the product of two commuting operators. The concerned problem aroused much interest in the 1970s and 1980s until the proposed conjecture was refuted by Müller and by Davidson and Holbrook right after.

Two of the generalized numerical ranges of the classical one, namely, the algebraic and essential numerical ranges, are the subjects of Chapter 4. The former can be defined generally for elements of a Banach algebra while the latter, as the special case for the Calkin algebra, is inherently connected with the compact operators. Two related classes of special operators, the commutator and zero-diagonal operators, and also the notion of total dilation are considered before a detailed study of the numerical ranges of compact operators is carried out.

A deep exploration of the relationships between the numerical range and the operator dilation is presented in Chapter 5. These include the classical unitary dilations of contractions, notable among which is the Choi–Li solution of a 35-year-old problem of Halmos concerning the numerical ranges of a contraction and its unitary dilations, the Berger power dilation for numerical contractions, and the power dilation of nilpotent contractions with excursions to the powerful Arveson extension theorem and Stinespring dilation theorem. The chapter concludes with spectral sets for operators, a notion originating from the von Neumann inequality for contractions. In recent years, the more general k-spectrality and complete k-spectrality attracted much attention due to a conjecture made by Crouzeix in 2004.

The next two chapters, Chapters 6 and 7, deal with numerical ranges of finite matrices. Here the extra tool of Kippenhahn polynomial comes in handy, which, among other things, is used to classify the numerical ranges of 3-by-3 matrices. Other effective devices are Bézout's theorem from algebraic geometry and the classical Riesz–Fejér theorem for trigonometric polynomials. These can be used to prove Anderson's theorem on circular numerical ranges of finite matrices and to estimate the number of line segments on the boundary of the numerical range. Nonnegative matrices, the ones with nonnegative entries, are the last topic considered in Chapter 6. Here Issos's result serves as the numerical range analogue of the Perron–Frobenius theorem for spectral properties of such matrices. It ends with discussions on the numerical ranges of doubly stochastic and row stochastic matrices.

Chapter 7 is on the numerical ranges of S_n-matrices. As the n-dimensional version of the more general compression of the shift, such matrices have numerical ranges enjoying many classical geometric properties such as the Poncelet porism and the generalized Brianchon–Ceva and Lucas–Siebeck theorems. An extension result of certain contractions to the direct sum of such operators via the Sz.-Nagy–Foiaş contraction theory provides more information on nilpotent operators in general and S_n-matrices in particular. The chapter ends with a result on the Gau–Wu numbers of S_n-matrices.

In the concluding Chapter 8, we consider various types of generalized numerical ranges other than those two discussed in Chapter 4. These include the joint numerical range, C-numerical range, q-numerical range, Davis–Wielandt shell, algebraic and spatial matricial ranges, and the higher-rank numerical range. Though some of them are not always convex, they all reveal rich theories in connection with the operators involved. The higher-rank numerical range developed in the past one and half decades is particularly interesting in this regard. The notion is motivated by the investigations in the quantum computing and quantum information theory, which are not covered in the present book. One pointer of its achievements is the Li–Sze characterization in 2008, which yields its convexity in a very natural way. The chapter ends in a discussion of the related notion of zero-dilation index of matrices.

At the end of each chapter, there is a collection of problems, numbered from the 40s to 70s. They are at diverse degrees of difficulty, all aiming at reinforcing in the reader's mind the ideas developed in the text. Some problems are just routine exercises filling up the missing links in the arguments for the proofs. Some give more details of special cases of the known results. Still others refer to results in the literature extending the relevant theory.

The Appendix after the main text collects some basic facts of convex sets in the real Euclidean space $\mathbb{R}^n$ and particularly in the plane $\mathbb{R}^2$ for the convenience of occasional references from the text.

Throughout the text, relevant comments and specific references have been given to each major (or minor) result following the flow of the narratives. This is against the usual practice in most books of collecting them at the end of each section or chapter. We believe that the present arrangement is more convenient for the reader to check the developments of each specific result without going back and forth.

A few words on our reference system. We refer to a theorem, proposition, lemma, corollary, or even a figure within each chapter by a two-number designation and from other chapters in three numbers. Thus, for example, Theorem 1.5 in Chapter 1 is referred to as Theorem 1.5 within Chapter 1 and as Theorem 1.1.5 from other chapters. In the same vein, Theorem I.1.1 and Theorem A.1.1 refer to Theorem 1.1 of the Introduction and of Appendix, respectively. In a multiple reference, we arrange the items contained therein in the chronological order. This is for the convenience of the reader as he or she goes through the literature. For example, in [164, 8], the reference [164] appeared years before [8] so that the results in [8] may depend on the ones in [164].

Throughout the years, the research interests of numerical ranges have been sustained mostly by the biennial "Workshop on Numerical Ranges and Numerical Radii" (WONRA). We have Chi-Kwong Li to thank for his tireless organization not only of this workshop but also many other yearly meetings and conferences in linear algebra. As one of the leaders of the Hong Kong school, he has made many fundamental contributions to the theory as were previously mentioned. We are very grateful to him and to the numerical range community in general, whose support through the years has given us the impetus to move forward. Thanks are also due to D. Farenick, whose incisive comments after a talk given by the first author in 1994 aroused the latter's initial interest in numerical ranges. Finally, the yearly grants from the Ministry of Science and Technology (formerly the National Science Council) of the Republic of China are gratefully acknowledged. The first author also thanks the Applied Math Department of National Chiao Tung University for its support over the past 45 years (even after his retirement).

List of Symbols

0_n	n-by-n zero matrix, 6
$0_{m,n}$	m-by-n zero matrix, 6
$[a_{ij}]_{i,j=1}^n$	n-by-n matrix, 2
$\#\triangle$	cardinality of $\triangle$, 1
$\aleph_0$	cardinality of $\mathbb{N}$ or $\mathbb{Z}$, 1
$\aleph_1$	cardinality of $\mathbb{R}$ or $\mathbb{C}$, 1
$\alpha(\theta)$	natural parametrization of $\partial\triangle$, 452
$\alpha(A)$	minimum of $\{i_{\geq 0}(\mathrm{Re}\,(e^{-i\theta}A)) : 0 \leq \theta < 2\pi\}$, 411
$\mathrm{Arg}\, z$	principal argument of z, 1
$\arg z$	argument of z, 1
$\triangle^*$	conjugate of $\triangle$, 3
$\triangle^\circ$	polar of $\triangle$, 246
$\triangle_\varepsilon$	open set $\{z \in \mathbb{C} : \mathrm{dist}\,(z, \triangle) < \varepsilon\}$, 25
$\bigvee M$	subspace spanned by M, 2
$\Delta(A)$	Aluthge transform of A, 29
$\Delta^{(n)}(A)$	nth-iterated Aluthge transform of A, 63
$\det A$	determinant of A, 6
$\mathrm{diag}\,(a_0, a_1, \ldots)$	diagonal operator with a_n's as diagonals, 5
$\mathrm{diam}\,\triangle$	diameter of $\triangle$, 39
$\dim H$	dimension of H, 1
$\mathrm{dist}\,(\lambda, \triangle)$	distance from λ to $\triangle$, 1
ℓ^2	space of square-summable sequences over $\mathbb{N}$, 2
$\ell^2(\mathbb{Z})$	space of square-summable sequences over $\mathbb{Z}$, 2
$\mathrm{Ext}\,\triangle$	set of extreme points of a convex set $\triangle$, 181
$\mathrm{Im}\, A$	imaginary part of A, 4
$\mathrm{ind}\, A$	Fredholm index of A, 9
$\inf \triangle$	infimum of $\triangle$, 1
$\mathrm{Int}\,\triangle$	(relative) interior of $\triangle$, 24
$\ker A$	kernel of A, 2
$\Lambda_k(A)$	rank-k numerical range of A, 406
$\lambda_k(A)$	kth-largest eigenvalue of Hermitian matrix A, 407

$\|\cdot\|$	norm, 1
$\|A\|$	norm of A, 2
$\langle\cdot,\cdot\rangle$	inner product, 1
$\langle \mathbf{A}x, y\rangle$	the vector $(\langle A_1x, y\rangle, \ldots, \langle A_mx, y\rangle)$ for $\mathbf{A} = (A_1, \ldots, A_m)$, 359
$\lceil t\rceil$	ceiling of t, 1
$\lfloor t\rfloor$	floor of t, 1
$\mathbb{C}$	complex numbers, 1
$\mathbb{CP}^2$	complex projective plane, 242
$\mathbb{D}$	open unit disc of $\mathbb{C}$, 1
$\mathbb{N}$	positive integers, 1
$\mathbb{R}$	real numbers, 1
$\mathbb{Z}$	integers, 1
$\mathcal{A}(\triangle)$	algebra of functions analytic on Int $\triangle$ and continuous on $\overline{\triangle}$, 220
$\mathcal{A}(\mathbb{D})$	disc algebra, 115
$\mathcal{B}(H)$	algebra of all operators on H, 2
$\mathcal{C}(H)$	Calkin algebra of H, 7
$\mathcal{C}_1$	trace-class operators, 6
$\mathcal{C}_2$	Hilbert–Schmidt operators, 6
$\mathcal{C}_p$	operators of Schatten p-class, 6
$\mathcal{F}_r$	operators F on H with rank $F \le r$, 420
$\mathcal{K}(H)$	compact operators on H, 9
$\mathcal{K}^{C^*\text{-}\wedge}$	C^*-convex hull of $\mathcal{K}$, 401
$\mathcal{P}_r$	orthogonal projections P on H with rank $P = r$, 420
$\mathcal{U}(A)$	unitary orbit of A, 371
$\mathcal{U}_n$	group of all n-by-n unitary matrices, 376
$\mathfrak{D}_A$	defect space of A, 196
max $\triangle$	maximum of $\triangle$, 1
$\text{mes}_2(\triangle)$	planar Lebesgue measure of $\triangle$, 50
min $\triangle$	minimum of $\triangle$, 1
$\Omega_k(A)$	set $\bigcap_{0\le\theta<2\pi} \{\lambda \in \mathbb{C} : \text{Re}\,(e^{-i\theta}\lambda) \le \lambda_k(\text{Re}\,(e^{-i\theta}A))\}$, 407
$\overline{x}$	vector of complex conjugates of components of x, 273
$\partial\triangle$	boundary of $\triangle$, 3
ran A	range of A, 2
rank A	rank of A, 10
Re A	real part of A, 3
$\rho(A)$	spectral radius of A, 3
$\rho_e(A)$	essential spectral radius of A, 161
$\rho_{\mathcal{A}}(x)$	spectral radius of x in $\mathcal{A}$, 148
$\sigma(A)$	spectrum of A, 3
$\sigma_c(A)$	continuous spectrum of A, 59
$\sigma_e(A)$	essential spectrum of A, 9
$\sigma_l(A)$	left spectrum of A, 3
$\sigma_p(A)$	point spectrum of A, 3

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$\sigma_r(A)$	right spectrum of A, 3
$\sigma_{\mathcal{A}}(x)$	spectrum of x in $\mathcal{A}$, 147
$\sigma_{ap}(A)$	approximate point spectrum of A, 3
$\sigma_{le}(A)$	left essential spectrum of A, 9
$\sigma_{re}(A)$	right essential spectrum of A, 9
$\sum_j \oplus A_j$	direct sum of A_j's, 2
$\sum_j \oplus H_j$	direct sum of H_j's, 2
$\sup \triangle$	supremum of $\triangle$, 1
$\mathrm{tr}\, A$	trace of A, 6
$\widehat{\triangle}$ or $\triangle^\wedge$	convex hull of $\triangle$, 15
$\widehat{A}$	Duggal transform of A, 28
$\lvert A\rvert$	$[\lvert a_{ij}\rvert]_{i,j=1}^n$ for $A = [a_{ij}]_{i,j=1}^n$, 274
$\lvert A\rvert$	positive square root of A^*A, 28
$\lvert x\rvert$	$(\lvert x_1\rvert, \ldots, \lvert x_n\rvert)$ for $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, 273
$\{A\}'$	commutant of A, 89
$\{A\}''$	double commutant of A, 305
$A \preccurlyeq B$	$B - A$ nonnegative, 274
$A(\triangle)$	area of $\triangle$, 453
$A < 0$	A negative definite, 4
$A > 0$	A positive definite, 4
$A[j_1, \ldots, j_m]$	principal submatrix of A indexed by $j_1, \ldots, j_m$, 296
$A \geq 0$	A positive semidefinite, 4
$A \leq 0$	A negative semidefinite, 4
$A \leq B$	$B - A$ positive semidefinite, 4
$A \otimes B$	tensor product of A and B, 131
$A \circ B$	Hadamard product of A and B, 356
$a \prec b$	a majorized by b, 354
$A \succcurlyeq 0_n$	A nonnegative, 6
$A\lvert K$	restriction of A to K, 4
A^*	adjoint of A, 2
A^+	positive part of Hermitian operator A, 167
A^-	negative part of Hermitian operator A, 167
$A^2(\mathbb{D})$	Bergman space on $\mathbb{D}$, 92
A^T	transpose of A, 6
$A^{(n)}$	inflation of A, 2
A^{-1}	inverse of A, 3
$A^{1/2}$	positive square root of A, 4
A_n	matrix $[a_{ij}]_{i,j=1}^n$ with $a_{ij} = 1$ for $j = i + 1$ or $(i, j) = (1, n)$ and 0 otherwise, 258
$A_n(1)$	matrix $[a_{ij}]_{i,j=1}^n$ with $a_{ij} = 1$ for $i < j$ and 0 otherwise, 330
$A_n(a)$	KMS matrix, 333
B	Bergman shift, 69
$B(\theta)$	breadth of $\triangle$ along the direction $(\cos\theta, \sin\theta)$, 452

B_n	matrix $[a_{ij}]_{i,j=1}^n$ with $a_{ij} = 1/n$ for all i and j, 278
$C(A)$	dual curve of $p_A(x, y, z) = 0$, 243
$C(A)$	generalized Crawford number of A, 33
$c(A)$	Crawford number of A, 33
$c(A)$	center of A, 37
$C^*(A)$	unital C^*-algebra generated by A, 397
$C_{\mathbb{R}}(A)$	Kippenhahn curve of A, 244
C_ϕ	composition operator, 74
$CP^n(A)$	class of unital completely positive maps from $C^*(A)$ to $M_n(\mathbb{C})$, 397
$d(\theta)$	support function, 450
$D(A)$	set $\{z \in \mathbb{C} : A - zI$ is zero-diagonal$\}$, 169
$d(A)$	zero-dilation index of A, 423
D_A	defect operator of A, 196
d_A	defect index of A, 196
$d_A(\theta)$	support function of $W(A)$, 182
$DW(A)$	Davis–Wielandt shell of A, 386
E_A	spectral measure of A, 57
$f_a(z)$	Blaschke factor, 115
F_σ	union of countably many closed sets, 83
G_A	associated undirected graph of A, 133
G_δ	intersection of countably many open sets, 82
$G_{\delta\sigma}$	union of countably many G_δ's, 82
$H(\phi)$	space $H^2 \ominus \phi H^2$, 305
$H \otimes K$	tensor product of H and K, 131
H^2	Hardy space, 2
H^∞	Hardy space of bounded analytic functions on $\mathbb{D}$, 10
H_n	space of n-by-n Hermitian matrices, 360
I	identity operator, 2
$i(\triangle)$	inradius of $\triangle$, 36
$i_+(A)$	number of positive eigenvalues of Hermitian matrix A, 423
$i_-(A)$	number of negative eigenvalues of Hermitian matrix A, 423
$i_0(A)$	number of zero eigenvalues of Hermitian matrix A, 423
I_H	identity operator on H, 2
I_n	n-by-n identity matrix, 6
$i_{\geq 0}(A)$	number of nonnegative eigenvalues of Hermitian matrix A, 423
$i_{\leq 0}(A)$	number of nonpositive eigenvalues of Hermitian matrix A, 423
j_A	norm-one index of A, 339
J_n	n-by-n Jordan block, 6
$J_n(\mu)$	n-by-n Jordan block with eigenvalue μ, 411
$k(A)$	Gau–Wu number of A, 345
$K^\perp$	orthogonal complement of K, 2
$K_1 \ominus K_2$	orthogonal difference of K_1 and K_2, 2

K_n	n-by-n weighted shift matrix with weights $\sqrt{2}, 1, \ldots, 1, \sqrt{2}, 0$, 124
$l(A)$	number of line segments on $\partial W(A)$, 262
$L^2(\mu)$	space of μ-square-integrable functions on X, 2
$L^2(\partial\mathbb{D})$	space of Lebesgue square-integrable functions on $\partial\mathbb{D}$, 2
$L^2[0,1]$	space of Lebesgue square-integrable functions on $[0,1]$, 2
$L^\infty(\mu)$	space of μ-essentially bounded functions on X, 58
$L^\infty(\partial\mathbb{D})$	space of Lebesgue essentially bounded functions on $\partial\mathbb{D}$, 65
$M(\triangle)$	maximal width of $\triangle$, 39
$m(\triangle)$	minimal width of $\triangle$, 39
$m(A)$	minimum modulus of A, 36
$m_A(z)$	minimal polynomial of A, 6
M_n or $M_n(\mathbb{C})$	space of n-by-n complex matrices, 6
M_ϕ	multiplication operator, 5
$n(\mathcal{A})$	numerical index of $\mathcal{A}$, 152
$nc(A)$	numerical center of A, 36
$p_A(x,y,z)$	Kippenhahn polynomial of A, 243
$p_A(z)$	characteristic polynomial of A, 6
P_K	(orthogonal) projection from H onto K, 4
$R(\triangle)$	outer radius of $\triangle$, 36
$r(\triangle)$	radius of $\triangle$, 453
R_ϕ	essential range of ϕ, 66
$S(\phi)$	compression of the shift on $H(\phi)$, 305
S_n	class of S_n-matrices, 10
S_n^{-1}	class of S_n^{-1}-matrices, 351
$t(A)$	transcendental radius of A, 37
T_ϕ	Toeplitz operator, 10
$W(\mathbf{A})$	numerical range of m-tuple $\mathbf{A} = (A_1, \ldots, A_m)$, 359
$W(A)$	numerical range of A, 11
$w(A)$	numerical radius of A, 32
$W(A_1, \ldots, A_m)$	joint numerical range of $A_1, \ldots, A_m$, 359
$W^n(A)$	algebraic matricial range of A, 397
$W_0(A)$	maximal numerical range of A, 37
$W_C(A)$	C-numerical range of A, 371
$W_c(A)$	c-numerical range of A, 371
$w_C(A)$	C-numerical radius of A, 439
$W_e(\mathbf{A})$	joint essential numerical range of $\mathbf{A} = (A_1, \ldots, A_m)$, 364
$W_e(A)$	essential numerical range of A, 155
$w_e(A)$	essential numerical radius of A, 161
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