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Tensor Products of  $C^*$ -Algebras and  
Operator Spaces  
The Connes–Kirchberg Problem

GILLES PISIER  
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University Printing House, Cambridge CB2 8BS, United Kingdom  
One Liberty Plaza, 20th Floor, New York, NY 10006, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,  
New Delhi – 110025, India  
79 Anson Road, #06–04/06, Singapore 079906

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Information on this title: [www.cambridge.org/9781108479011](http://www.cambridge.org/9781108479011)

DOI: 10.1017/9781108782081

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First published 2020

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

*A catalogue record for this publication is available from the British Library.*

ISBN 978-1-108-47901-1 Hardback

ISBN 978-1-108-74911-4 Paperback

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## Contents

<b>Introduction</b>	<i>page 1</i>
<b>1 Completely bounded and completely positive maps: Basics</b>	11
1.1 Completely bounded maps on operator spaces	11
1.2 Extension property of $B(H)$	18
1.3 Completely positive maps	23
1.4 Normal c.p. maps on von Neumann algebras	30
1.5 Injective operator algebras	31
1.6 Factorization of completely bounded (c.b.) maps	33
1.7 Normal c.b. maps on von Neumann algebras	37
1.8 Notes and remarks	39
<b>2 Completely bounded and completely positive maps: A tool kit</b>	41
2.1 Rows and columns: operator Cauchy–Schwarz inequality	41
2.2 Automatic complete boundedness	43
2.3 Complex conjugation	44
2.4 Operator space dual	48
2.5 Bi-infinite matrices with operator entries	50
2.6 Free products of $C^*$ -algebras	53
2.7 Universal $C^*$ -algebra of an operator space	57
2.8 Completely positive perturbations of completely bounded maps	58
2.9 Notes and remarks	61
<b>3 <math>C^*</math>-algebras of discrete groups</b>	63
3.1 Full (=Maximal) group $C^*$ -algebras	63
3.2 Full $C^*$ -algebras for free groups	66
3.3 Reduced group $C^*$ -algebras: Fell’s absorption principle	71
3.4 Multipliers	73
3.5 Group von Neumann Algebra	77

3.6	Amenable groups	78
3.7	Operator space spanned by the free generators in $C_\lambda^*(\mathbb{F}_n)$	83
3.8	Free products of groups	84
3.9	Notes and remarks	85
<b>4</b>	<b><math>C^*</math>-tensor products</b>	87
4.1	$C^*$ -norms on tensor products	87
4.2	Nuclear $C^*$ -algebras (a brief preliminary introduction)	91
4.3	Tensor products of group $C^*$ -algebras	92
4.4	A brief repertoire of examples from group $C^*$ -algebras	95
4.5	States on the maximal tensor product	96
4.6	States on the minimal tensor product	99
4.7	Tensor product with a quotient $C^*$ -algebra	103
4.8	Notes and remarks	104
<b>5</b>	<b>Multiplicative domains of c.p. maps</b>	106
5.1	Multiplicative domains	106
5.2	Jordan multiplicative domains	108
5.3	Notes and remarks	112
<b>6</b>	<b>Decomposable maps</b>	113
6.1	The dec-norm	113
6.2	The $\delta$ -norm	121
6.3	Decomposable extension property	125
6.4	Examples of decomposable maps	129
6.5	Notes and remarks	135
<b>7</b>	<b>Tensorizing maps and functorial properties</b>	136
7.1	$(\alpha \rightarrow \beta)$ -tensorizing linear maps	136
7.2	$\ \cdot\ _{\max}$ is projective (i.e. exact) but not injective	141
7.3	max-injective inclusions	144
7.4	$\ \cdot\ _{\min}$ is injective but not projective (i.e. not exact)	150
7.5	min-projective surjections	153
7.6	Generating new $C^*$ -norms from old ones	157
7.7	Notes and remarks	160
<b>8</b>	<b>Biduals, injective von Neumann algebras, and <math>C^*</math>-norms</b>	161
8.1	Biduals of $C^*$ -algebras	161
8.2	The nor-norm and the bin-norm	162
8.3	Nuclearity and injective von Neumann algebras	163
8.4	Local reflexivity of the maximal tensor product	170
8.5	Local reflexivity	174
8.6	Notes and remarks	179

## Contents

vii

<b>9</b>	<b>Nuclear pairs, WEP, LLP, QWEP</b>	180
9.1	The fundamental nuclear pair $(C^*(\mathbb{F}_\infty), B(\ell_2))$	181
9.2	$C^*(\mathbb{F})$ is residually finite dimensional	186
9.3	WEP (Weak Expectation Property)	188
9.4	LLP (Local Lifting Property)	193
9.5	To lift or not to lift (global lifting)	198
9.6	Linear maps with WEP or LLP	202
9.7	QWEP	204
9.8	Notes and remarks	208
<b>10</b>	<b>Exactness and nuclearity</b>	210
10.1	The importance of being exact	210
10.2	Nuclearity, exactness, approximation properties	216
10.3	More on nuclearity and approximation properties	222
10.4	Notes and remarks	224
<b>11</b>	<b>Traces and ultraproducts</b>	225
11.1	Traces	225
11.2	Tracial probability spaces and the space $L_1(\tau)$	228
11.3	The space $L_2(\tau)$	230
11.4	An example from free probability: semicircular and circular systems	235
11.5	Ultraproducts	238
11.6	Factorization through $B(H)$ and ultraproducts	246
11.7	Hypertraces and injectivity	256
11.8	The factorization property for discrete groups	259
11.9	Notes and remarks	261
<b>12</b>	<b>The Connes embedding problem</b>	262
12.1	Connes's question	262
12.2	The approximately finite dimensional (i.e. "hyperfinite") $II_1$ -factor	269
12.3	Hyperlinear groups	271
12.4	Residually finite groups and Sofic groups	273
12.5	Random matrix models	276
12.6	Characterization of nuclear von Neumann algebras	277
12.7	Notes and remarks	279
<b>13</b>	<b>Kirchberg's conjecture</b>	280
13.1	LLP $\Rightarrow$ WEP?	280
13.2	Connection with Grothendieck's theorem	283
13.3	Notes and remarks	290

<b>14</b>	<b>Equivalence of the two main questions</b>	291
14.1	From Connes's question to Kirchberg's conjecture	291
14.2	From Kirchberg's conjecture to Connes's question	292
14.3	Notes and remarks	296
<b>15</b>	<b>Equivalence with finite representability conjecture</b>	297
15.1	Finite representability conjecture	297
15.2	Notes and remarks	299
<b>16</b>	<b>Equivalence with Tsirelson's problem</b>	300
16.1	Unitary correlation matrices	300
16.2	Correlation matrices with projection valued measures	303
16.3	Strong Kirchberg conjecture	309
16.4	Notes and remarks	310
<b>17</b>	<b>Property (T) and residually finite groups: Thom's example</b>	311
17.1	Notes and remarks	316
<b>18</b>	<b>The WEP does not imply the LLP</b>	317
18.1	The constant $C(n)$ : $\text{WEP} \not\Rightarrow \text{LLP}$	319
18.2	Proof that $C(n) = 2\sqrt{n-1}$ using random unitary matrices	323
18.3	Exactness is not preserved by extensions	327
18.4	A continuum of $C^*$ -norms on $\mathbb{B} \otimes \mathbb{B}$	329
18.5	Notes and remarks	332
<b>19</b>	<b>Other proofs that <math>C(n) &lt; n</math>: quantum expanders</b>	333
19.1	Quantum coding sequences. Expanders. Spectral gap	333
19.2	Quantum expanders	336
19.3	Property (T)	338
19.4	Quantum spherical codes	341
19.5	Notes and remarks	343
<b>20</b>	<b>Local embeddability into <math>\mathcal{C}</math> and nonseparability of <math>(OS_n, d_{cb})</math></b>	344
20.1	Perturbations of operator spaces	345
20.2	Finite-dimensional subspaces of $\mathcal{C}$	346
20.3	Nonseparability of the metric space $OS_n$ of $n$ -dimensional operator spaces	351
20.4	Notes and remarks	357
<b>21</b>	<b>WEP as an extension property</b>	358
21.1	WEP as a local extension property	358
21.2	WEP versus approximate injectivity	362
21.3	The (global) lifting property LP	364
21.4	Notes and remarks	365



<b>22</b>	<b>Complex interpolation and maximal tensor product</b>	366
22.1	Complex interpolation	366
22.2	Complex interpolation, WEP and maximal tensor product	371
22.3	Notes and remarks	382
<b>23</b>	<b>Haagerup's characterizations of the WEP</b>	384
23.1	Reduction to the $\sigma$ -finite case	384
23.2	A new characterization of generalized weak expectations and the WEP	385
23.3	A second characterization of the WEP and its consequences	388
23.4	Preliminaries on self-polar forms	390
23.5	$\max^+$ -injective inclusions and the WEP	395
23.6	Complement	403
23.7	Notes and remarks	408
<b>24</b>	<b>Full crossed products and failure of WEP for <math>\mathcal{B} \otimes_{\min} \mathcal{B}</math></b>	410
24.1	Full crossed products	410
24.2	Full crossed products with inner actions	414
24.3	$\mathcal{B} \otimes_{\min} \mathcal{B}$ fails WEP	418
24.4	Proof that $C_0(3) < 3$ (Selberg's spectral bound)	427
24.5	Other proofs that $C_0(n) < n$	429
24.6	Random permutations	431
24.7	Notes and remarks	432
<b>25</b>	<b>Open problems</b>	434
	<b>Appendix: Miscellaneous background</b>	438
A.1	Banach space tensor products	438
A.2	A criterion for an extension property	439
A.3	Uniform convexity of Hilbert space	441
A.4	Ultrafilters	441
A.5	Ultraproducts of Banach spaces	443
A.6	Finite representability	443
A.7	Weak and weak* topologies: biduals of Banach spaces	444
A.8	The local reflexivity principle	446
A.9	A variant of Hahn–Banach theorem	447
A.10	The trace class	448
A.11	$C^*$ -algebras: basic facts	448
A.12	Commutative $C^*$ -algebras	450
A.13	States and the GNS construction	451
A.14	On $*$ -homomorphisms	452
A.15	Approximate units, ideals, and quotient $C^*$ -algebras	454
A.16	von Neumann algebras and their preduals	456

A.17	Bitransposition: biduals of $C^*$ -algebras	461
A.18	Isomorphisms between von Neumann algebras	465
A.19	Tensor product of von Neumann algebras	466
A.20	On $\sigma$ -finite (countably decomposable) von Neumann algebras	466
A.21	Schur's lemma	467
	<i>References</i>	470
	<i>Index</i>	482