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978-1-108-47901-1 — Tensor Products of C*-Algebras and Operator Spaces
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More Information

Introduction

These lecture notes are centered around two open problems, one formulated by Alain Connes in his famous 1976 paper [61], the other one by Eberhard Kirchberg in his landmark 1993 paper [155]. At first glance, these two problems seem quite different and the proof of their equivalence described at the end of [155] is not so easy to follow. One of our main goals is to explain in detail the proof of this equivalence in an essentially self-contained way. The Connes problem asks roughly whether traces on "abstract" von Neumann algebras can always be approximated (in a suitable way) by ordinary matrix traces. The Kirchberg problem asks whether there is a unique C^*-norm on the algebraic tensor product $\mathscr{C} \otimes \mathscr{C}$ when $\mathscr{C}$ is the full C^*-algebra of the free group $\mathbb{F}_\infty$ with countably many generators.

In the remarkable paper where he proved the equivalence, Kirchberg studied more generally the pairs of C^*-algebras (A, B) for which there is only one C^*-norm on the algebraic tensor product $A \otimes B$. We call such pairs "nuclear pairs." A C^*-algebra A is traditionally called nuclear if this holds for any C^*-algebra B. Our exposition chooses as its cornerstone Kirchberg's theorem asserting the nuclearity of what is for us the "fundamental pair," namely the pair $(\mathscr{B}, \mathscr{C})$ where $\mathscr{B} = B(\ell_2)$ (see Theorem 9.6). Our presentation leads us to highlight two properties of C^*-algebras, the Weak Expectation Property (WEP) and the Local Lifting Property (LLP).

The first one is a weak sort of extension property (or injectivity) while the second one is a weak sort of lifting property. The connection with the fundamental pair is very clear: A has the WEP (resp. LLP) if and only if the pair $(A, \mathscr{C})$ (resp. $(A, \mathscr{B})$) is nuclear. With this terminology, the Kirchberg problem reduces to proving the implication LLP $\Rightarrow$ WEP, but there are many more interesting reformulations that deserve mention and we will present them in detail. For instance this problem is equivalent to the question whether *every* (unital) C^*-algebra is a quotient of one with the WEP, or equivalently, in short, is QWEP. In passing, although the P stands for property, we will sometimes write for short that A is WEP (or A is LLP) instead of A has the WEP (resp. LLP).

Incidentally, since Kirchberg (unlike Connes) explicitly conjectured a positive answer to all these equivalent questions in [155], we often refer to them as his conjectures.

One originality of our treatment (although already present in [155]) is that we try to underline the structural properties (or their failure), such as injectivity or projectivity, in parallel for the minimal and the maximal tensor product of C^*-algebras. This preoccupation can be traced back to the "fundamental pair" itself: Indeed, we may view $\mathscr{B}$ as "injectively universal" and $\mathscr{C}$ as "projectively universal." The former because any separable C^*-algebra A is a subalgebra of $\mathscr{B}$, the latter because any such A is a quotient of $\mathscr{C}$ (see Proposition 3.39).

In particular, we will emphasize the fact that the minimal tensor product is injective but not projective, while the maximal one is projective but not injective (see §7.4 and 7.2). This is analogous to the situation that prevails for the Banach space tensor products in Grothendieck's classical work, but unlike Banach space morphisms (i.e. bounded linear maps) the C^*-algebraic morphisms are automatically isometric if they are injective (see Proposition A.24). The lack of injectivity of the max-norm is a rephrasing of the fact that if $B_1 \subset B_2$ is an isometric (or equivalently injective) $*$-homomorphism between C^*-algebras and A is another C^*-algebra, it is in general *not true* that the resulting $*$-homomorphism

$$A \otimes_{\max} B_1 \to A \otimes_{\max} B_2 \tag{1}$$

is isometric (or equivalently injective). This means that the norm induced by $A \otimes_{\max} B_2$ on the algebraic tensor product $A \otimes B_1$ is *not* equivalent to the max-norm on $A \otimes B_1$. In sharp contrast, this does not happen for the min-norm: $A \otimes_{\min} B_1 \to A \otimes_{\min} B_2$ is always injective (=isometric), and this is why one often says that the minimal tensor product is "injective."

This "defect" of the max-tensor product leads us to single out the class of inclusions, $B_1 \subset B_2$, for which this defect disappears (i.e. (1) is injective for any A). We choose to call them "max-injective." We will see that this holds if and only if there is a projection $P : B_2^{**} \to B_1^{**}$ with $\|P\| = 1$. We will also show that if (1) is injective for $A = \mathscr{C}$ then it is injective for all A.

It turns out that a C^*-algebra A is WEP if and only if the embedding $A \subset B(H)$ is max-injective or, equivalently, if and only if there is a projection $P : B(H)^{**} \to A^{**}$ with $\|P\| = 1$. All these facts have analogues for the min-tensor product, but now its "defect" is the failure of "projectivity," meant in the following sense: Let $q : B_1 \to B_2$ be a surjective $*$-homomorphism and let A be any C^*-algebra. Let $\mathcal{I} = \ker(q)$. Then, although the associated

$*$-homomorphism $q_A : A \otimes_{\min} B_1 \to A \otimes_{\min} B_2$ is clearly surjective (indeed, it suffices for that to have a dense range), its kernel may be strictly larger than $A \otimes_{\min} \mathcal{I}$. As a result, the min-norm on the algebraic tensor product $A \otimes B_2$ $(= A \otimes (B_1/\mathcal{I}))$ may be much smaller than the norm induced on it by $(A \otimes_{\min} B_1)/(A \otimes_{\min} \mathcal{I})$. In sharp contrast, this "defect" does not happen for the max-norm and we always have an isometric identification

$$A \otimes_{\max} (B_1/\mathcal{I}) = (A \otimes_{\max} B_1)/(A \otimes_{\max} \mathcal{I}).$$

Again this defect of the min-norm leads us to single out the quotient maps (i.e. the surjective $*$-homomorphisms) $q : B_1 \to B_2$ for which the defect does not appear, i.e. the maps such that for any A we have an isometry

$$A \otimes_{\min} B_2 = (A \otimes_{\min} B_1)/(A \otimes_{\min} \mathcal{I}). \tag{2}$$

Here again, we can give a rather neat characterization of such maps, this time as a certain form of lifting property, see §7.5. It turns out that if (2) holds for $A = \mathscr{B}$ then it holds for all C^*-algebras A. We call such a map q a "min-projective surjection." The usual terminology to express that (2) holds for any A is that B_1 viewed as an extension of B_2 by $\mathcal{I}$ is a "locally split extension" (we prefer not to use this term). This notion is closely connected with the notion of exact C^*-algebra.

A C^*-algebra A is called exact if (2) holds for any surjective $q : B_1 \to B_2$. This "exact" terminology is motivated by the fact that (2) holds if and only if the sequence

$$0 \to A \otimes_{\min} \mathcal{I} \to A \otimes_{\min} B_1 \to A \otimes_{\min} B_2 \to 0$$

is exact. But actually, for C^*-algebras, the exactness of that sequence boils down to the fact that the natural $*$-homomorphism

$$\frac{A \otimes_{\min} B_1}{A \otimes_{\min} \mathcal{I}} \to A \otimes_{\min} B_2$$

is isometric (=injective).

Although our main interest is in C^*-algebras, it turns out that many results have better formulations (and sometimes better proofs) when phrased using linear subspaces of C^*-algebras (the so-called operator spaces) or unital self-adjoint subspaces (the so-called operator systems). It is thus natural to try to describe as best as we can the class of linear transformations that preserve the C^*-tensor products. For the minimal norm, it is well known that the associated class is that of completely bounded (c.b.) maps. More precisely, given a linear map $u : A \to B$ between C^*-algebras we have for any C^*-algebra C

$$\|Id_C \otimes u : C \otimes_{\min} A \to C \otimes_{\min} B\| \le \|u\|_{cb} \tag{3}$$

where $\|u\|_{cb}$ is the c.b. norm of u. Moreover, the sup over all C of the left-hand side of (3) is equal to $\|u\|_{cb}$, and it remains unchanged when restricted to $C \in \{M_n \mid n \geq 1\}$. The space of such maps is denoted by $CB(A, B)$.

The mapping u is called completely positive (in short c.p.) if $Id_C \otimes u : C \otimes_{\min} A \to C \otimes_{\min} B$ is positive (=positivity preserving) for any C, and to verify this we may restrict to $C = M_n$ for any $n \geq 1$. The cone formed of all such maps is denoted by $CP(A, B)$.

For the max tensor product, there is an analogue of (3) but the corresponding class of mappings is smaller than $CB(A, B)$. These are the decomposable maps denoted by $D(A, B)$, defined as linear combinations of maps in $CP(A, B)$. More precisely, for any u as previously we have

$$\|Id_C \otimes u : C \otimes_{\max} A \to C \otimes_{\max} B\| \leq \|u\|_{dec}, \tag{4}$$

where $\|u\|_{dec}$ is the norm in $D(A, B)$. Moreover, the supremum over all C of the left-hand side of (4) is equal to the dec-norm of u composed with the inclusion $B \subset B^{**}$. The dec-norm was introduced by Haagerup in [104]. We make crucial use of several of the properties established by him in the latter paper. See Chapter 6.

The third class of maps that we analyze are the maps $u : A \to B$ such that for any C

$$\|Id_C \otimes u : C \otimes_{\min} A \to C \otimes_{\max} B\| \leq 1.$$

This holds if and only if u is the pointwise limit of a net of finite rank maps with $\|u\|_{dec} \leq 1$ (see Proposition 6.13). When u is the identity on A this means that A has the c.p. approximation property (CPAP) which, as is by now well known, characterizes nuclear C^*-algebras (see Corollary 7.12).

More generally, suppose given two C^*-norms α and β, defined on $A \otimes B$ for any pair (A, B). We denote by $A \otimes_\alpha B$ (resp. $A \otimes_\beta B$) the C^*-algebra obtained after completion of $A \otimes B$ equipped with α (resp. β).

Then we say that a linear map $u : A \to B$ between C^*-algebras is $(\alpha \to \beta)$-tensorizing if for any C^*-algebra C

$$\|Id_C \otimes u : C \otimes_\alpha A \to C \otimes_\beta B\| \leq 1.$$

In §7.1 we describe the factorizations characterizing such maps in all the cases when α and β are either the minimal or the maximal C^*-norm. We also include the case when u is only defined on a subspace $E \subset A$ using the norm induced on $C \otimes E$ by $C \otimes_\alpha A$. The main cases of interest are min $\to$ max (nuclearity) and max $\to$ max (decomposability). For the former, we refer to Chapter 10, where we characterize nuclear C^*-algebras in parallel with exactness.

The bidual A^{**} of a C^*-algebra A is isomorphic to a von Neumann algebra. In Chapter 8 we study the relations between C^*-norms on A and on A^{**}

and we describe the biduals of certain C^*-tensor products. The notion of local reflexivity plays an important role in that respect. We prove in §8.3 the equivalence of the injectivity of A^{**} and the nuclearity of A. In Corollary 7.12 (proved in §10.2) we show that for C^*-algebras nuclearity is equivalent to the completely positive approximation property (CPAP). We also show in Theorem 8.12 that injective von Neumann algebras are characterized by a weak* analogue of the CPAP, which is sometimes called "semidiscreteness."

But our main emphasis is on *nuclear pairs*: in §9.1 we prove the nuclearity of the fundamental pair $(\mathscr{B},\mathscr{C})$ and in the rest of Chapter 9 we give various equivalent characterizations of C^*-algebras with the properties WEP, LLP, and QWEP, that we choose to define using nuclear pairs. The main ones are formulated using the bidual A^{**} of a C^*-algebra A (see §8.1). Let $i_A : A \to A^{**}$ be the natural inclusion. For instance:

(i) A is nuclear if and only if for some (or any) embedding $A^{**} \subset B(H)$ there is a projection $P : B(H) \to A^{**}$ with $\|P\|_{cb} = 1$.
(ii) A is WEP if and only if for some (or any) embedding $A \subset B(H)$ there is a projection $P : B(H)^{**} \to A^{**}$ with $\|P\|_{cb} = 1$.
(iii) A is QWEP if and only if for some embedding $A^{**} \subset B(H)^{**}$ there is a projection $P : B(H)^{**} \to A^{**}$ with $\|P\|_{cb} = 1$.

We then come to the central part of these notes: the Connes embedding problem whether any tracial probability space embeds in an ultraproduct of matricial ones (Chapter 12) and the Kirchberg conjecture (Chapter 13) that $\mathscr{C}$ is WEP or that every C^*-algebra is QWEP. We show that they are equivalent in Chapter 14. We also show the equivalence with a well-known conjecture from Banach space theory (Chapter 15). The latter essentially asserts that every von Neumann algebra is isometric (as a Banach space) to a quotient of $B(H)$ for some H. In yet another direction we show in Chapter 16 that all these conjectures are equivalent to one formulated by Tsirelson in the context of quantum information theory.

In one of its many equivalent forms, Kirchberg's conjecture reduces to LLP $\Rightarrow$ WEP for C^*-algebras. Actually, he originally conjectured also the converse implication but in Chapter 18 we show that this fails, by producing tensors $t \in \mathscr{B} \otimes \mathscr{B}$ for which the min and max norms are different; in other words the pair $(\mathscr{B},\mathscr{B})$ is not nuclear. The proof combines ideas from finite-dimensional operator space theory (indeed $t \in E \otimes F$ for some finite-dimensional subspaces E, F of $\mathscr{B}$) together with estimates of spectral gaps, that allow us to show that a certain constant $C(n)$ defined next is $<n$ for some n. The latter constant involves a sequence of integers N_m and a sequence $(u_1(m), \dots, u_n(m))$ of n-tuples of unitary $N_m \times N_m$-matrices and their complex conjugates $(\overline{u_1(m)}, \dots, \overline{u_n(m)})$. We then set

$$C(n) = \inf\left\{ \sup_{m \neq m'} \left\| \sum\nolimits_1^n \overline{u_j(m)} \otimes u_j(m') \right\| \right\}, \tag{5}$$

where the last norm is meant in $M_{N_m N_{m'}}$ and the infimum runs over all possible sizes (N_m) and all possible sequences $(u_1(m), \ldots, u_n(m))$ of n-tuples of unitary $N_m \times N_m$-matrices.

Using unitary random matrices we will show that $C(n) = 2\sqrt{n-1}$ (see §18.2). Nevertheless other more explicit (deterministic) constructions of sequences $(u_1(m), \ldots, u_n(m))$ responsible for $C(n) < n$ are of much interest such as property (T) groups, expanders, quantum expanders, or quantum analogues of spherical coding sequences. In each case we obtain a tensor $t \in \mathscr{B} \otimes \mathscr{B}$ such that $\|t\|_{\min} < \|t\|_{\max}$. We describe these delicate ingredients in Chapter 19.

In Chapter 20, we gather several applications of the preceding ideas to the structure of the metric space of all finite-dimensional operator spaces equipped with the cb-analogue of the Banach–Mazur “distance,” that is defined when $\dim(E) = \dim(F)$ by

$$d_{cb}(E, F) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb} \mid u : E \to F \text{ invertible}\}.$$

For instance, for any finite-dimensional operator space E, the dual space E^* admits a natural operator space structure (described in §2.4) so that we may view both E and E^* as subspaces of $\mathscr{B}$. Thus the identity operator on E defines a tensor $t_E \in \mathscr{B} \otimes \mathscr{B}$. We show that (see (20.6))

$$\|t_E\|_{\mathscr{B} \otimes_{\max} \mathscr{B}} = \inf\{d_{cb}(E, F) \mid F \subset \mathscr{C}\}$$

where the infimum (which is actually attained) runs over all possible subspaces $F \subset \mathscr{C}$ with $\dim(F) = \dim(E)$.

The fact that $(\mathscr{B}, \mathscr{B})$ is not a nuclear pair actually implies that for arbitrary von Neumann algebras (M, N) the pair (M, N) is nuclear only if either M or N is nuclear. This follows from the fact that a nonnuclear von Neumann algebra must contain as a subalgebra the direct sum in the sense of ℓ_∞ of the family $\{M_n \mid n \geq 1\}$ of all matrix algebras, and there is automatically a conditional expectation onto it. The latter is explained in §12.6.

In Chapter 23 we present in detail two unpublished characterizations of the WEP due to Haagerup. The first one says that a C^*-algebra A has the WEP if and only if for any n and any linear map $u : \ell_\infty^n \to A$ the dec-norm of u coincides with its c.b. norm (see §23.2). This naturally complements his earlier results from the 1980s in [104]. Haagerup claimed this theorem at some point in the 1990s but apparently did not circulate a detailed proof of it, as he did for the second (more delicate) one, that we give in §23.5.

There, to put it very roughly ℓ_∞^n is replaced by ℓ_2^n. More precisely, the second characterization says that A has the WEP if and only if for any n and any $(a_1,\ldots,a_n) \in A^n$ we have

$$\left\|\sum \overline{a_j} \otimes a_j\right\|_{\min}^{1/2} = \left\|\sum \overline{a_j} \otimes a_j\right\|_{\max}^{1/2}.$$

An important ingredient for its proof is the identification, for any C^*-algebra A, of the norm

$$A^n \ni (a_j) \mapsto \left\|\sum \overline{a_j} \otimes a_j\right\|_{\max}^{1/2}$$

as the norm obtained on A^n $(n \geq 1)$ by the complex interpolation method of parameter $\theta = 1/2$ between the ("row and column") norms

$$(a_j) \mapsto \left\|\sum a_j{}^* a_j\right\|^{1/2} \text{ and } (a_j) \mapsto \left\|\sum a_j a_j{}^*\right\|^{1/2}.$$

In order to give a reasonably self-complete proof of the latter fact we give a brief basic description of complex interpolation in Chapter 22.

One important consequence of this particular characterization is the fact that the WEP is stable under complete isomorphisms. Explicitly, if two C^*-algebras A, B are completely isomorphic as operator spaces, then A WEP $\Rightarrow$ B WEP. In other words, if we forget the algebraic structure of a C^*-algebra, the WEP is "remembered" by its operator space structure.

In a similar flavor (see Chapter 23), let $M \subset \mathcal{M}$ be von Neumann algebras, if there is a completely bounded projection $P : \mathcal{M} \to M$ onto M (i.e. M is "completely complemented" in $\mathcal{M}$) then there is a projection $Q : \mathcal{M} \to M$ that is completely positive with $\|Q\|_{cb} = 1$. Thus when $\mathcal{M} = B(H)$ we conclude that M is injective.

In Chapter 24 we show that the tensor product $M \otimes_{\min} N$ of two nonnuclear von Neumann algebras M and N (for instance for $M = N = \mathscr{B}$) fails the WEP (see Corollary 24.23). The proof is reminiscent of the earlier proof that (M, N) is not a nuclear pair. It makes crucial use of the constant that we denote by $C_0(n)$, that is defined in the same way as $C(n)$ in (5), but using unitaries associated to permutations instead of plain unitary matrices and restricting them to the orthogonal of the constant vector. Again, the key point is that $C_0(n) < n$. We review the recent results that establish the latter. In analogy with the case of $C(n)$ we can show that $C_0(n) = 2\sqrt{n-1}$ using a very recent result on random permutation matrices, and also that $C_0(3) < 3$ by delicate deterministic arguments: we can use either Selberg's famous spectral bound or known results on expanders in permutation groups.

Lastly, in Chapter 25 we gather a collection of open questions related to our main topics.

Prerequisites. These notes are written in a rather detailed style and should be accessible to graduate students and nonspecialists. The prerequisite background is kept to a minimum. Of course basic functional analysis is needed, but for operator algebras, the fundamental theorems we use are the classical ones, such as the bicommutant theorem and Kaplansky's Theorem, as well as basic facts about states, $*$-homomorphisms and the GNS construction, and we review all those in this book's Appendix.

Sources. The main source for these notes is Kirchberg's fundamental paper [155]. However, we have made extensive use of Haagerup's treatment of decomposable maps in [104]. This allowed us to reformulate many results known for completely positive maps or $*$-homomorphisms for just *linear* maps. In addition, Ozawa's surveys [189, 191, 192] have been an invaluable help and inspiration, as well as the (highly recommended) book [39] by Brown and Ozawa.

Many of Kirchberg's results on exactness are already presented in detail in Simon Wassermann's excellent 1994 notes [258], the present volume can be viewed as a sequel and an updated complement to his.

Almost all chapters are followed by a Notes and Remarks section where we try to complement the references given in the text, and sometimes add some pointers to the literature.

About operator spaces. Some results already appear in our 2003 book on operator spaces [208]. When convenient, we used the presentation from [208]. We describe several applications of operator space theory when they are relevant for our topic, but our *main focus* being here on tensor products of C^*-*algebras*, we will refrain from developing operator space theory for its own sake, and we refer the reader instead to [208], or to [80, 196].

About operator systems. Following Arveson's pioneering papers [12], much work (notably by Choi, Effros, and Lance) on operator systems appeared already in the 1970s which marked a first period when much progress on tensor products of C^*-algebras was achieved. In particular, Choi and Effros introduced in [47] a notion of duality for operator systems that prefigured the one for operator spaces developed after Ruan's 1987 Ph.D. thesis. The emphasis then moved on to operator spaces in the 1990s, and C^*-tensor products were investigated (following Kirchberg's impulse and Haagerup's work) in the more general framework of operator space tensor products, by Effros, Ruan, Blecher, Paulsen, and others. Curiously, operator systems made a reappearance more recently and their tensor products were investigated thoroughly in a series of papers, notably [150, 151]. This led to several characterizations of the WEP (see [90–92, 149, 152, 153]), connected to the Connes–Kirchberg problem, but for lack of space (and energy) we chose not to cover this.

We also had to leave out the connections of the Connes–Kirchberg problem with noncommutative real algebraic geometry, for which we refer the reader to [40, 163] and to Ozawa's survey [191].

Basic notation and conventions. The letter H (or $\mathcal{H}$) always stands for a Hilbert space. Our Hilbert spaces all have an inner product

$$(y,x) \mapsto \langle y,x\rangle$$

that is linear in x and antilinear in y.

We denote by $B(H)$ (resp. $K(H)$) the Banach algebra formed of all the bounded (resp. compact) linear operators on H equipped with the operator norm.

Let K be another Hilbert space. We denote by $K \otimes_2 H$ the Hilbert space tensor product, obtained by completing $K \otimes H$ equipped with the classical scalar product characterized by

$$\langle k \otimes h, k' \otimes h'\rangle = \langle k,k'\rangle\langle h,h'\rangle.$$

We denote by $\overline{K}$ the complex conjugate Hilbert space, which is classically identified with the dual K^*. Then $\overline{K} \otimes_2 H$ can be identified with the space of all the Hilbert–Schmidt maps from K to H.

The unitary group of a unital C^*-algebra A is denoted by $U(A)$.

The identity map on a linear space X is denoted by Id_X.

The unit ball of a normed space X is denoted by B_X.

Let $1 \le p \le \infty$. Let I be an arbitrary index set. We denote by $\ell_p(I)$ the set of families of complex scalars $x = (x_i)_{i\in I}$ such that $\sum_{i\in I} |x_i|^p < \infty$ ($\sup_{i\in I} |x_i| < \infty$ when $p = \infty$) equipped with the norm $\|x\|_p = (\sum_{i\in I} |x_i|^p)^{1/p}$ ($\sup_{i\in I} |x_i|$ when $p = \infty$).

When $I = \mathbb{N}$, we denote $\ell_p(I)$ simply by ℓ_p.

Let $(X_i)_{i\in I}$ be a family of Banach spaces. We denote by

$$\left(\oplus \sum\nolimits_{i\in I} X_i\right)_p$$

their direct sum "in the sense of ℓ_p," equipped with the norm $(x_i) \mapsto (\sum \|x_i\|^p)^{1/p}$.

When $X_i = X$ for all $i \in I$, we denote $\left(\oplus \sum_{i\in I} X_i\right)_p$ by $\ell_p(I;X)$.

When $X_i = \mathbb{C}$ for all $i \in I$ we recover $\ell_p(I)$.

In the particular case when $p = \infty$ the space $\mathcal{X} = \left(\oplus \sum_{i\in I} X_i\right)_\infty$ is the set of those $x = (x_i)$ with $x_i \in X_i$ ($\forall i \in I$) such that $\|x\| = \sup_{i\in I} \|x_i\| < \infty$.

The unit ball of this space $\mathcal{X}$ is just the product $B_{\mathcal{X}} = \prod_{i\in I} B_{X_i}$.

Let $n \geq 1$ be an integer. We denote by ℓ_p^n the space $\mathbb{C}^n$ equipped with the norm

$$x \mapsto \|x\| = \left(\sum\nolimits_1^n |x_j|^p\right)^{1/p}.$$

Thus $\ell_p^n = \ell_p(I)$ for $I = \{1, \ldots, n\}$. When $p = 2$, the resulting space ℓ_2^n is the model for any n-dimensional Hilbert space.

When $p = \infty$, we set $\|x\| = \sup_j |x_j|$, the resulting space ℓ_∞^n is the model for any n-dimensional commutative C^*-algebra.

We denote by M_n (resp. $M_{n\times m}$) the space of $n \times n$ (resp. $n \times m$) matrices with complex entries. More generally, for any vector space E we will denote by $M_n(E)$ (resp. $M_{n\times m}(E)$) the space of $n \times n$ (resp. $n \times m$) matrices with entries in E. Thus $M_n = M_n(\mathbb{C})$ (resp. $M_{n\times m} = M_{n\times m}(\mathbb{C})$).

A linear mapping $u : X \to Y$ between Banach spaces with $\|u\| \leq 1$ is called "contractive" or "a contraction." We say that $u : X \to Y$ is a metric surjection if $u(X) = Y$ and the image of the open unit ball of X coincides with the open unit ball of Y. Then passing to the quotient by $\ker(u)$ produces an isometric isomorphism from $X/\ker(u)$ to Y.

A mapping $u : A \to B$ between C^*-algebras is called a $*$-homomorphism if it is a homomorphism of algebras such that $u(x^*) = u(x)^*$ for all $x \in A$. When $B = B(H)$ for some Hilbert space H the term "representation" is often used instead of $*$-homomorphism.

Some abbreviations frequently used. c.b. for completely bounded, c.p. for completely positive, u.c.p. for unital and completely positive, c.c. for completely contractive.

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