MATRIX POSITIVITY

Matrix positivity is a central topic in matrix theory: properties that generalize the notion of positivity to matrices arose from a large variety of applications, and many have also taken on notable theoretical significance, either because they are natural or unifying or because they support strong implications. This is the first book to provide a comprehensive and up-to-date reference of important material on matrix positivity classes, their properties, and their relations. The matrix classes emphasized in this book include the classes of semipositive matrices, P-matrices, inverse M-matrices, and copositive matrices.

This self-contained reference will be useful to a large variety of mathematicians, engineers, and social scientists, as well as graduate students. The generalizations of positivity and the connections observed provide a unique perspective, along with theoretical insight into applications and future challenges. Direct applications can be found in data analysis, differential equations, mathematical programming, computational complexity, models of the economy, population biology, dynamical systems, and control theory.

CHARLES R. JOHNSON is Class of 1961 Professor of Mathematics at College of William and Mary. He received his Ph.D. from The California Institute of Technology in 1972. Professor Johnson has published nearly 500 papers and 12 books and received several prizes.

RONALD L. SMITH is Professor Emeritus at the University of Tennessee, Chattanooga. He received his Ph.D. in Mathematics at Auburn University. He received the Thomson-Reuters Original List of Highly Cited Researchers Award. The conference “Recent Advances in Linear Algebra and Graph Theory” was held in his honor.

MICHAEL J. TSATSOMEROS is Professor of Mathematics at Washington State University. He received his Ph.D. in Mathematics at the University of Connecticut in 1990. He has served on the Board of Directors and Advisory and Journal Committees of the International Linear Algebra Society. He is co-Editor-in-Chief of the *Electronic Journal of Linear Algebra* and Associate Editor of *Linear Algebra and Its Applications*.

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Matrix Positivity

CHARLES R. JOHNSON
College of William and Mary

RONALD L. SMITH
University of Tennessee at Chattanooga

MICHAEL J. TSATSOMEROS
Washington State University

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To the many students who have helped me with mathematics through my REU program at William and Mary. They have made it fun for me as well as enhanced greatly what I have been able to do.
Charles R. Johnson

To my parents and siblings; to Linda, my wife and best friend, whose help on this book was invaluable to me; and to Kevin, Brad, Reid, and Sarah, our children who gave us six wonderful grandchildren.
Ronald L. Smith

To Μόσχα, Katerina, and Joanne. Your inspiration and support make it all possible.
Michael J. Tsatsomeros

Contents

Preface

Properties that generalize the notion of positivity (of scalars) to matrices have arisen from a large variety of applications. Many have also taken on notable theoretical significance, either because they are natural or unifying or because they support strong implications. All three authors have written extensively on matricial generalizations of positivity, over a long period of time. Some of these generalizations are already the subject of one or more books, sometimes with excellent and modern treatments. But, for a variety of reasons, others are not yet treated comprehensively in readily available form. We feel that a good reference for these will be a useful contribution. This is the purpose of the present work.

After a review of relevant background in Chapter 1, we discuss the most prominent generalizations of positivity in Chapter 2. The containment and other relationships among these are given, as well as useful book references for some. It is shown that all of them are contained among the "semipositive" matrices. There are, of course, variations upon these prominent generalizations too, and there are often weak and strong versions of the generalizations.

In subsequent chapters, particular generalizations of positivity are studied in detail, when we feel that we can make significant and novel contributions. These include semipositive matrices in Chapter 3, P-matrices in Chapter 4, Inverse M-matrices in Chapter 5, and copositive matrices in Chapter 6. Then we conclude with a lengthy list of references for these areas.

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Symbols

- $\mathbb{R}$, $\mathbb{C}$ Fields of real, complex numbers
- $\mathbb{R}^n$, $\mathbb{C}^n$ Column n-vectors of real, complex numbers
- $\mathbb{R}^n_+$ Nonnegative orthant in $\mathbb{R}^n$ (all n-vectors of nonnegative numbers)
- $M_{m,n}(\mathbb{F})$ The m-by-n matrices over field $\mathbb{F}$; skipping $\mathbb{F}$ means $\mathbb{F} = \mathbb{C}$
- $M_n(\mathbb{F}) = M_{n,n}(\mathbb{F}),\ M_n = M_n(\mathbb{C}) = M_{n,n}(\mathbb{C})$
- $M_n(\{-1, 0, 1\})$ The n-by-n matrices with entries in $\{-1, 0, 1\}$
- $M_n(\{-1, 1\})$ The n-by-n matrices with entries in $\{-1, 1\}$
- X^T, X^* Transpose and conjugate transpose of a complex array X
- $X^\dagger$ Moore–Penrose inverse of $X \in M_{m,n}(\mathbb{C})$
- $X^\#$ Group inverse of $X \in M_{m,n}(\mathbb{C})$
- $X > Y$ ($X \geq Y$) Real arrays X, Y, every entry of $X - Y$ is positive (nonnegative)
- $[X, Y]$ Matrix interval ($Y \geq X$), all real matrices Z such that $X \leq Z \leq Y$
- $\langle n \rangle = \{1, 2, \ldots, n\}$.
- $A \circ B$ Hadamard (entry-wise) product of $A, B \in M_{m,n}(\mathbb{F})$
- $A^{(k)}$ k-th Hadamard power $A \circ A \circ \cdots \circ A$, $A \in M_{m,n}(\mathbb{F})$
- index(A) Index of A
- Δ_n Unit simplex in $\mathbb{R}^n$
- J All ones square matrix
- e All ones column vector
- $\mathrm{Tr}(A)$ Trace of $A \in M_n(\mathbb{F})$
- $Q(x) = x^T Ax$ Associated quadratic form, $A \in M_n(\mathbb{C})$, $x \in \mathbb{C}^n$
- adj A Adjoint of $A \in M_{m,n}(\mathbb{C})$
- $R(A)$ Range of $A \in M_{m,n}(\mathbb{C})$
- rank(A) Rank of $A \in M_{m,n}(\mathbb{C})$
- Null(A) (Right) null space of $A \in M_{m,n}(\mathbb{C})$
- nullity(A) Dimension of Null(A)
- $\sigma(A)$ Spectrum (eigenvalues) of $A \in M_n(\mathbb{C})$

- $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ Spectral radius of $A \in M_n(\mathbb{C})$
- $q(A)$ Positive eigenvalue of minimum modulus of an M-matrix A
- $\operatorname{diag}(d_1, d_2, \ldots, d_n)$ The n-by-n diagonal matrix with diagonal entries $d_1, d_2 \ldots, d_n$
- $|\alpha|$ Cardinality of $\alpha \subseteq \langle n \rangle$
- $\alpha^c = \langle n \rangle \setminus \alpha$ Complement of $\alpha \subseteq \langle n \rangle$ in $\langle n \rangle$
- $A[\alpha, \beta]$ Submatrix of A lying in rows $\alpha \subseteq \langle m \rangle$ and columns $\beta \subseteq \langle n \rangle$
- $A[\alpha] = A[\alpha, \alpha]$ Principal submatrix of A lying in rows $\alpha \subseteq \langle n \rangle$
- $A(\alpha, \beta) = A[\alpha^c, \beta^c]$
- $A(i) = A(\{i\})$
- $A/A[\alpha]$ Schur complement of $A[\alpha]$ in $A \in M_n(\mathbb{F})$, $\alpha \subseteq \langle n \rangle$
- ppt (A, α) Principal pivot transform of $A \in M_n(\mathbb{C})$ relative to $\alpha \subseteq \langle n \rangle$
- F_A Cayley transform of $A \in M_n(\mathbb{C})$
- $\mathcal{P}(A/A[\alpha])$ Perron complement of $A[\alpha]$ in $A \in M_n(\mathbb{F})$, $\alpha \subseteq \langle n \rangle$
- $S_+(A) = \{x \in \mathbb{R}^n : x \geq 0 \text{ and } Ax > 0\}$, $A \in M_{m,n}(\mathbb{R})$
- $R_+(A) = A\,S_+(A)$, $A \in M_{m,n}(\mathbb{R})$
- $K_+(A) = \{x \in \mathbb{R}^n : x \geq 0 \text{ and } Ax \geq 0\}$, $A \in M_{m,n}(\mathbb{R})$
- $A \oslash B = \begin{bmatrix} A \\ B \end{bmatrix} \in M_{m+p,n}(\mathbb{R}), \quad A \in M_{m,n}(\mathbb{R})$ and $B \in M_{p,n}(\mathbb{R})$
- $A \ominus B = [A\ B] \in \mathrm{M}_{m,n+q}, \quad A \in M_{m,n}(\mathbb{R})$ and $B \in M_{m,q}(\mathbb{R})$
- $\mathcal{M}(A)$ Comparison matrix of $A \in M_n(\mathbb{C})$
- $D(A)$ Directed graph of $A \in M_n(\mathbb{C})$
- $S(A)$ Signed directed graph of $A \in M_n(\mathbb{R})$
- LCP (q, M) The Linear Complementarity Problem, $M \in M_n(\mathbb{R})$, $q \in \mathbb{R}^n$
- $m(A)$ Measure of irreducibility of $A \in M_n(\mathbb{C})$
- $U(A)$ Upper path product bound for $A \in M_n(\mathbb{C})$
- $I(A, B)$ Interval from $A \in M_n(\mathbb{R})$ to $B \in M_n(\mathbb{R})$
- $V(A, B)$ Vertex matrices derived from $A, B \in M_n(\mathbb{R})$

Matrix Classes

- **C** ($\mathbf{C}_n$) Copositive matrices (n-by-n)
- **CP** Completely positive matrices
- D_n n-by-n invertible diagonal matrices
- D_n^+ n-by-n positive diagonal matrices in D_n
- **DN** Doubly nonnegative matrices
- **DP** Doubly positive matrices
- **EIM** Eventually inverse M-matrices
- **EP** (Entry-wise) positive matrices
- **IM** Inverse M-matrices

- **IM**^{D} Dual of **IM**
- **IS** Identically signed class of matrices
- **LSP** Left semipositive matrices
- M_n^k $A \in M_n(\mathbb{R})$ with nonzero diagonal entries and length of longest cycle in $D(A) \leq k$
- **M** M-matrices
- **MSP** Minimally semipositive matrices
- **N** (Entry-wise) Nonnegative matrices
- $\mathbf{N}^+$ Nonnegative matrices with positive main diagonal
- **ND** Negative definite matrices
- **NSD** Negative semidefinite matrices
- P_n Group of n-by-n permutation matrices
- P_n^k Set of real P-matrices in M_n^k
- **P** P-matrices
- $\mathbf{P}_{\mathrm{M}}$ Matrices all of whose powers are P-matrices
- $\mathbf{P}_0$ P_0-matrices
- **PD** Positive definite matrices
- **PSD** Positive semidefinite matrices
- **PSPP** Purely strict path product matrices
- **PP** Path product matrices
- **RSP** Redundantly semipositive matrices
- $\mathbf{S}_n = \mathbf{S}_n(\mathbb{R})$ Set of symmetric matrices in $M_n(\mathbb{R})$
- S_n^k Set of $A \in M_n(\mathbb{R})$ all of whose cycles in $S(-A)$ are signed negatively
- **SC** ($\mathbf{SC}_n$) Strictly copositive matrices (n-by-n)
- **SIM** Symmetric inverse M-matrices
- **sN** Symmetric nonnegative matrices
- **SN** Seminegative matrices
- **SNN** Seminonnegative matrices
- **SNP** Seminonpositive matrices
- **SP** Semipositive matrices
- **SPP** Strict path product matrices
- **SZ** Semizero matrices
- **TN** Totally nonnegative matrices
- **TP** Totally positive matrices
- **TSPP** Totally strict path product matrices
- **Z** Z-matrices