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For Rayna and Dylan

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Preface

I first encountered the Assouad dimension on Wednesday, 20 April 2011 whilst attending an EPSRC workshop on Dynamical Systems and Dimension hosted by the University of Warwick. James Robinson gave a talk entitled Assouad Dimension, Cube Slicing, and the Dynamics on Finite-Dimensional Attractors, which I remember included a discussion of the fact that the maximal volume of the intersection of the unit cube in $\mathbb{R}^d$ with an affine hyperplane is $\sqrt{2}$ (counter-intuitively independent of d). I was intrigued by the Assouad dimension, and surprised that I had not heard of it before, especially given my interest in the box and Hausdorff dimensions. Following the talk, I found two papers on the topic, one by Lars Olsen (2011), which gave a direct proof of the fact that self-similar sets satisfying the open set condition have equal Hausdorff and Assouad dimensions, and one by Mackay (2011) which computed the Assouad dimension of certain self-affine sets. Coincidentally, both of these papers were also published in the same year. Another significant paper on the topic, which I found shortly after the papers of Mackay and Olsen, is an article by Luukkainen (1998). This article established many of the basic properties of the Assouad dimension, but its main focus was in proving a 'Szpilrajn Theorem' for Assouad dimension: the topological dimension of a separable metric space X is the infimum of the Assouad dimension of metric spaces Y such that X and Y are homeomorphic. Szpilrajn (1937) proved this with the Assouad dimension replaced by the Hausdorff dimension.[1]

Olsen posed two (classically fractal) questions concerning the Assouad dimension, which at the time lay only at the fringes of the fractal geometry dialogue, see Olsen (2011, questions 1.3 and 1.4). The first question asked if the Assouad and Hausdorff dimensions could be distinct for self-similar

[1] Szpilrajn later changed his name to Marczewski to avoid persecution in Nazi Germany.

sets (obviously requiring failure of the open set condition) and the second asked for the Assouad dimension of the famous self-affine sets introduced by Bedford (1984) and McMullen (1984). The second of these questions was answered in Mackay's paper and I was able to answer the first one (in the affirmative) (Fraser 2014). All I needed was an example, which I will describe later in this book in Theorem 7.2.1, but this example led to a fruitful collaboration with James Robinson, Eric Olson, and Alexander Henderson, in which we precisely described the Assouad dimension of all self-similar sets in $\mathbb{R}$, see Theorem 7.2.4. This work was completed in 2014 and published the following year (Fraser et al. 2015). I was hooked and since then I have spent a lot of time investigating the Assouad dimension, learning about its subtle, often counter-intuitive, properties and exploring its connections with dimension theory, fractal geometry, and wider mathematics.

The Assouad dimension is rapidly becoming part of the mainstream dialogue in fractal geometry and the need for this book is highlighted by the fact that it is not mentioned in some of the most important and influential books in the field. For example, it is not mentioned in the books by Falconer (1985a, 1997, 2014, 2013b), Mattila (1995, 2015), Bishop and Peres (2017), Edgar (1990), Mandelbrot (1982), or Barnsley (1988, 2006). In many ways I see this book as a companion to, for example, Kenneth Falconer's *Fractal Geometry: Mathematical Foundations and Applications* (2014), which presents a detailed and comprehensive treatment of dimension theory from a fractal geometry perspective. Many of the examples and problems I consider will have a similar flavour to those in Falconer (2014).

The Assouad dimension is mentioned in – and is a central focus of – some books closely related to fractal geometry. In particular, *Dimensions, Embeddings, and Attractors* by Robinson (2011), and *Conformal Dimension: Theory and Application* by Mackay and Tyson (2010). Robinson's book focuses on the general embedding theory of dynamical systems, where the Assouad dimension is a key technical tool with many applications. We will discuss some examples in this direction in Section 13.3. Mackay and Tyson's book focuses on the problem of lowering dimension (usually Hausdorff or Assouad) via quasi-symmetric maps. This is a subtle and challenging problem and the Assouad dimension highlights many key features of this exploration. We will touch on this area in Chapter 12. Despite the important role played by dimension theory, these books (Robinson 2011; Mackay and Tyson 2010) do not have what I would call *classical fractal geometry* at their heart, but rather encounter fractal ideas on a journey motivated by a different set of problems. As such, this book serves a different purpose and will consider more *classically fractal* questions in the context of the Assouad dimension, such as the dimension theory of iterated functions systems (self-similar, self-affine sets, other dynamical invariants) and geometric constructions (projections, products, slices, distance sets). As well as developing the theory in the context of fractal geometry, I will also explore numerous applications to problems in areas such as embedding theory, arithmetic geometry, Diophantine approximation, probability theory, and functional analysis. There will inevitably be some overlap with Robinson (2011) and Mackay and Tyson (2010) – for example, Mackay and Tyson's work on weak tangents and Assouad dimension has been central to recent progress in fractal geometry – but I will attempt to keep repetition to a minimum. Another purpose of this book is to consider various natural variations on the Assouad dimension, including its natural dual the lower dimension, the quasi-Assouad dimension, the Assouad spectrum (which I introduced with my first PhD student, Han Yu), and the corresponding dimensions of measures. I aim to present these ideas in a unified way and to inspire future explorations in these directions. As such, this book also contains several new results as I attempt to present a unified and comprehensive theory.

The book is not meant to be a comprehensive survey and I apologise in advance for omitting a detailed discussion of much interesting work. Where appropriate I have tried to include a large set of references and to indicate where more details could have been included. Often, results are not presented in their most general form. For example, I almost exclusively restrict my attention to sets and measures in Euclidean space despite most of the theory applying in more general metric spaces. My goal here is to present the key ideas in as simple a framework as possible. Again, where possible, I will indicate where more general results can be found. Elementary real analysis will be assumed and some familiarity with fractal geometry and dimension theory would be beneficial, but should not be required. Some of my favourite introductory books on analysis include: Howie (2001), Rudin (1976), and Stewart and Tall (1997).

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