PART ONE

Theory

1

Fractal Geometry and Dimension Theory

In this introductory chapter we briefly discuss the history and development of fractal geometry and dimension theory. We introduce and motivate some important concepts such as Hausdorff and box dimension. As part of this discussion we encounter covers and packings, which are central notions in dimension theory, and introduce the dimension theory of measures.

1.1 The Emergence of Fractal Geometry

A *fractal* can be described as an object which exhibits interesting features on a large range of scales; see Figure 1.1. In pure mathematics, the Sierpiński triangle, the middle third Cantor set, the boundary of the Mandelbrot set, and the von Koch snowflake are archetypal examples and, in ‘real life’, examples include the surface of a lung, the horizon of a forest, and the distribution of stars in the galaxy. The fractal story began in the nineteenth century with the appearance of a multitude of strange examples exhibiting what we now understand as fractal phenomena. These included Weierstrass’ example of a continuous nowhere differentiable function, Cantor’s construction of an uncountable set with zero length, and Brown’s observations on the path taken by a piece of pollen suspended in water (Brownian motion). During the first half of the twentieth century the mathematical foundations for fractal geometry were laid down by, for example, Besicovitch, Bouligand, Hausdorff, Julia, Marstrand, and Sierpiński, and the theory was unified and popularised by the extensive writings of Mandelbrot in the 1970s, for example Mandelbrot (1982). It was Mandelbrot who coined the term ‘fractal’, derived from the Latin *fractus* meaning ‘broken’. Since then the subject has grown and developed as a self-contained discipline in pure mathematics touching on many other subjects, such as dynamical systems, geometric measure theory, analysis (real and complex), topology, number theory, probability theory, and harmonic analysis. However, the importance of fractals is not restricted to abstract mathematics, with many naturally occurring physical phenomena exhibiting a fractal structure, such as graphs of random processes, percolation models, and fluid turbulence. The mathematical challenge is to understand the mechanisms which generate and underpin fractal behaviour and to develop robust and rich theories concerning the geometric properties that such objects possess.

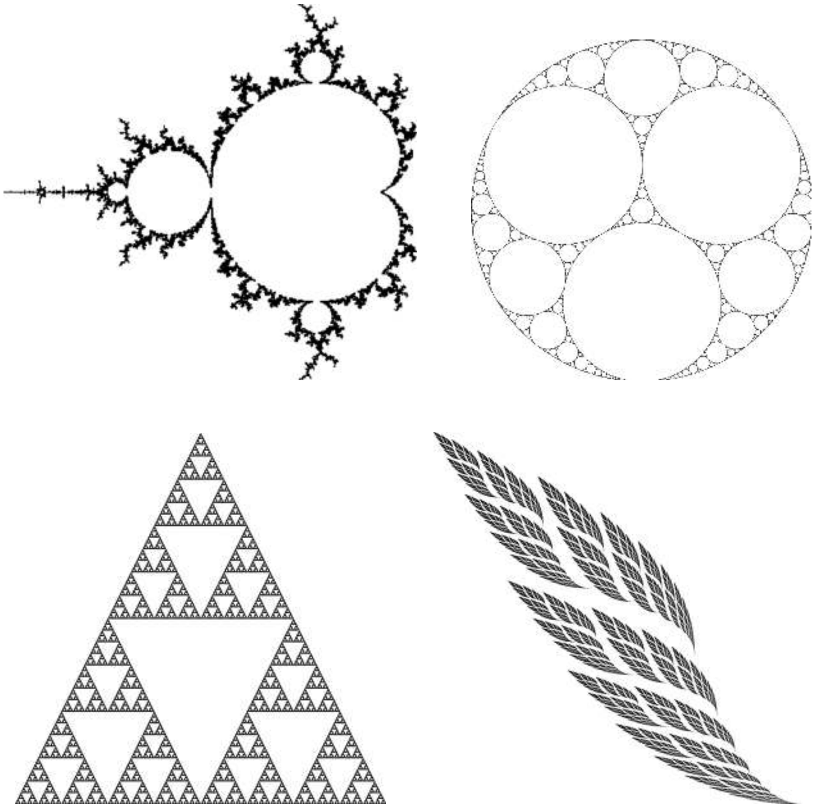

Figure 1.1 *Four fractals.* From top left moving clockwise: the boundary of the Mandelbrot set, the Apollonian circle packing, a (self-affine) leaf, and the Sierpiński triangle.

1.2 Dimension Theory

At the heart of fractal geometry lies *dimension theory*, the subject dedicated to understanding how to define, interpret, understand, and calculate dimensions of sets in Euclidean space or more general metric spaces. A *dimension* is a (usually non-negative real) number which gives geometric information concerning how the object in question fills up space on small scales. There are many distinct notions of dimension and one of the joys (and central components) of the subject is in understanding how these notions relate to each other, and how their behaviour compares in different settings or when applied to different families of examples.

A natural approach to dimension theory is to quantify how large a set is at a given scale by considering optimal *covers* by balls whose diameter is related to the scale. More precisely, given a scale $r > 0$, a finite or countable collection of sets $\{U_i\}_i$ is called an *r-cover* of a set F if each of the sets U_i has diameter less than or equal to r, and F is contained in the union $\bigcup_i U_i$; see Figure 1.2. Throughout the book we write $|U| = \sup_{x,y \in U} |x - y|$ for the diameter of a non-empty set $U \subseteq \mathbb{R}^d$. Understanding how to find covers of a set at small scales underpins much of dimension theory and often the 'covering strategy' is specific to the setting, sometimes driven by dynamical invariance or a priori knowledge of another, related, set. This book is dedicated to a thorough analysis of the Assouad dimension and some of its natural variants. However, we will often attempt to put our discussion in a wider context for which we require other notions.

The Hausdorff dimension is arguably the most well-studied and important notion of fractal dimension. It was introduced by Hausdorff (1918), greatly developed by Besicovitch, and is considered extensively in many of the important books on fractal geometry, such as Bishop and Peres (2017); Falconer (1997, 2014); Mattila (1995). It is defined in terms of Hausdorff measure, which can be viewed as a natural extension of Lebesgue measure to non-integer dimensions. Given $s \geqslant 0$ and $r > 0$, the *r-approximate s-dimensional Hausdorff measure* of a set $F \subseteq \mathbb{R}^d$ is defined by

$$\mathcal{H}^s_r(F) = \inf\left\{\sum_i |U_i|^s : \{U_i\}_i \text{ is a countable } r\text{-cover of } F\right\}$$

and the *s-dimensional Hausdorff (outer) measure* of F is $\mathcal{H}^s(F) = \lim_{r \to 0} \mathcal{H}^s_r(F)$. The limit exists because the sequence $\mathcal{H}^s_r(F)$ increases as r decreases, but it may be infinite. The measures $\mathcal{H}^s$ can now be used to identify the *critical exponent* or *dimension* in which it is most appropriate to consider F. First, consider the square $[0,1]^2$, which has infinite length (length measures objects which are much smaller, such as line segments or smooth curves), and zero volume (volume measures objects which are much bigger, such as cubes or spheres). However, the *area* of the square is positive and finite (it hardly matters that the precise area is 1), demonstrating that the natural measure to use when considering squares is area, that is, 2-dimensional Lebesgue measure. It is no coincidence that we think of the square as a 2-dimensional object. Since we have continuum many Hausdorff measures to choose from, this leads naturally to the *Hausdorff dimension* of F being defined as

$$\dim_{\mathrm{H}} F = \inf\left\{s \geqslant 0 \colon \mathcal{H}^s(F) = 0\right\} = \sup\left\{s \geqslant 0 \colon \mathcal{H}^s(F) = \infty\right\}.$$

It is a useful exercise to show that these two expressions for the Hausdorff dimension actually coincide. The value of the Hausdorff measure in the critical dimension, that is, $\mathcal{H}^{\dim_{\mathrm{H}} F}(F)$, is often rather hard to compute exactly and can be any value in $[0, \infty]$.

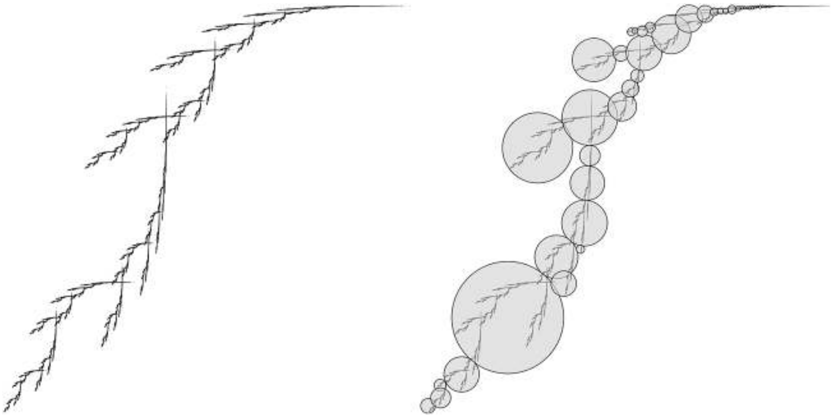

Figure 1.2 Left: a self-affine fractal. Right: a covering of the self-affine fractal using balls of arbitrarily varying radii. Understanding such covers leads to calculation of the Hausdorff dimension.

A less sophisticated, but nevertheless very useful, notion of dimension is box dimension. The *lower and upper box dimensions* of a non-empty bounded set $F \subseteq \mathbb{R}^d$ are defined by

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{r \to 0} \frac{\log N_r(F)}{-\log r} \qquad \text{and} \qquad \overline{\dim}_{\mathrm{B}} F = \limsup_{r \to 0} \frac{\log N_r(F)}{-\log r},$$

respectively, where $N_r(F)$ is the smallest number of open sets required for an r-cover of F; see Figure 1.3. If $\underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F$, then we call the common value the *box dimension* of F and denote it by $\dim_{\mathrm{B}} F$. Note that, unlike the

Hausdorff dimension, the box dimension is usually only defined for bounded sets since $N_r(F) = \infty$ for any unbounded set.

Notice that for a bounded set F, $r > 0$ and $s \geqslant 0$,

$$\mathcal{H}_r^s(F) \leqslant r^s N_r(F),$$

which immediately gives $\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F$.

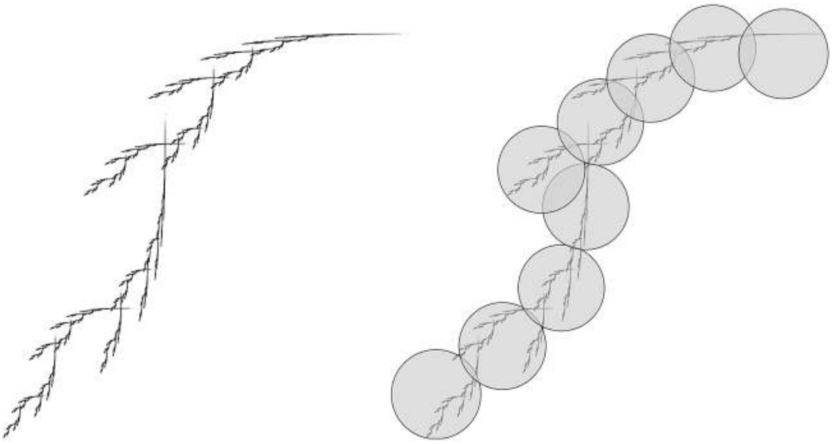

Figure 1.3 Left: the self-affine set from Figure 1.2. Right: a covering using balls of constant radii. Understanding such covers leads to calculation of the box dimensions.

One final notion, which we will mention less frequently, is the packing dimension. This can be defined by a suitable modification of the upper box dimension designed to make it countably stable; see Section 2.4. For $F \subseteq \mathbb{R}^d$, the *packing dimension* of F is defined by

$$\dim_{\mathrm{P}} F = \inf \left\{ \sup_i \overline{\dim}_{\mathrm{B}} F_i : \ F = \bigcup_i F_i \right\}. \tag{1.1}$$

This definition works perfectly well for unbounded sets if we assume the F_i are bounded. Moreover, one immediately gets $\dim_{\mathrm{H}} F \leqslant \dim_{\mathrm{P}} F \leqslant \overline{\dim}_{\mathrm{B}} F$. The packing dimension turns out to be a natural 'dual notion' to the Hausdorff dimension, where packings are used instead of covers. The usual formulation of packing dimension first defines packing measure, as a dual to the Hausdorff measure, and then packing dimension in the natural way. It was first introduced by Claude Tricot (1982).

We write $B(x,r) = \{y \in \mathbb{R}^d : |y - x| \leqslant r\}$ to denote the closed ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$. A collection of balls $\{B(x_i, r)\}_i$ is called a (centred) r-*packing* of a set F if the balls are pairwise disjoint and, for all i, $x_i \in F$. A related notion is that of r-separated sets. A set $X \subseteq F$ is called an r-*separated* subset of F if each pair of distinct points from X are separated by a distance of at least r. If $r_i = r$ for all i, then the set of centres $\{x_i\}_i$ of balls from an r-packing form a $2r$-separated subset of F. It is an elementary but instructive exercise to prove that if one replaces $N_r(F)$ in the definition of upper and lower box dimensions with any of

(i) the maximum number of balls in an r-packing of F,
(ii) the maximum cardinality of an r-separated subset of F,
(iii) the number of r-cubes in an axes oriented mesh which intersect F, or
(iv) the minimum number of cubes of sidelength r required to cover F

then one obtains the same values for the box dimensions; see Figure 1.4. See Falconer (2014, section 2.1) for a more detailed discussion of this, along with some direct calculations.

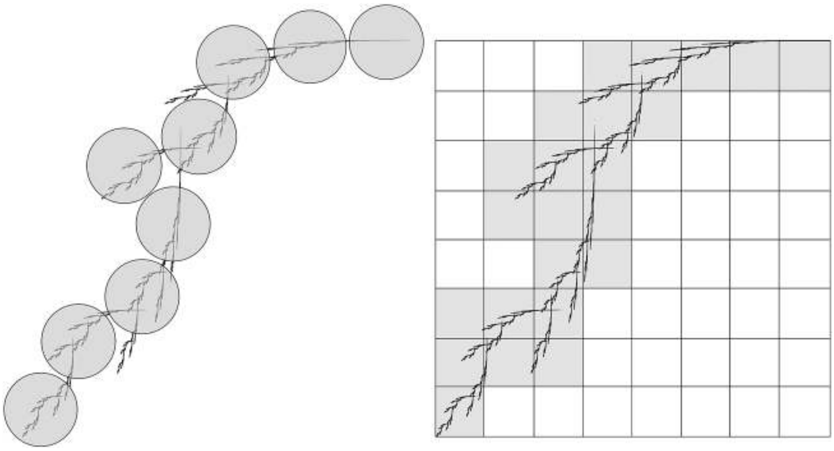

Figure 1.4 A packing of the self-affine set from Figure 1.2 using balls of constant radii centred in F (left) and a mesh of squares imposed on F with the squares intersecting F shown in grey (right).

1.2.1 Dimension Theory of Measures

An important aspect of dimension theory is the interplay between the dimensions of sets and the dimensions of measures; see, for example, the mass distribution principle, Lemma 3.4.2. For this we need analogous notions of dimension for measures. The *(lower) Hausdorff dimension* of a Borel measure μ on $\mathbb{R}^d$ is

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} E : \mu(E) > 0\}.$$

Thus the dimension of a measure is conveniently expressible in terms of dimensions of sets which 'see' the measure. We write

$$\mathrm{supp}(\mu) = \{x \in \mathbb{R}^d : \ \mu(B(x,r)) > 0 \text{ for all } r > 0\} \tag{1.2}$$

for the *support* of μ, which is necessarily a closed set, and we say μ is *fully supported* on $F \subseteq \mathbb{R}^d$ if $\mathrm{supp}(\mu) = F$ and *supported* on $F \subseteq \mathbb{R}^d$ if $\mathrm{supp}(\mu) \subseteq F$. Straight from the definition one has, for μ supported on F,

$$\dim_{\mathrm{H}} \mu \leqslant \dim_{\mathrm{H}} F.$$

In fact, for Borel sets F,

$$\dim_{\mathrm{H}} F = \sup\{\dim_{\mathrm{H}} \mu : \ \mathrm{supp}(\mu) \subseteq F\}. \tag{1.3}$$

This follows by finding compact subsets $E \subseteq F$ with positive and finite s-dimensional Hausdorff measure for all $s < \dim_{\mathrm{H}} F$, see Falconer (2014, theorem 4.10). Similarly, the *(lower) packing dimension* of a Borel measure μ on $\mathbb{R}^d$ is

$$\dim_{\mathrm{P}} \mu = \inf\{\dim_{\mathrm{P}} E : \mu(E) > 0\}.$$

Again one has, for μ supported on F,

$$\dim_{\mathrm{P}} \mu \leqslant \dim_{\mathrm{P}} F$$

and, for Borel sets F,

$$\dim_{\mathrm{P}} F = \sup\{\dim_{\mathrm{P}} \mu : \ \mathrm{supp}(\mu) \subseteq F\}.$$

This final result follows by a similar approach, this time due to Joyce and Preiss (1995). The box dimension of a measure is a less well-developed concept. We formulate a definition in Section 4.2 following Falconer et al. (2020), which is partially motivated by the Assouad spectrum; see Section 3.3.

2

The Assouad Dimension

In this chapter we define the Assouad dimension, which is the central notion of the book. We discuss its origins in Section 2.3 and establish many of its basic properties in Section 2.4 such as stability under Lipschitz mappings and monotonicity. These are compared with the basic properties of the Hausdorff and box dimensions.

2.1 The Assouad Dimension and a Simple Example

If the Hausdorff dimension provides fine, but global, geometric information, then the Assouad dimension provides coarse, but local, geometric information. The key difference between the Assouad dimension and the dimensions discussed in Chapter 1 is that only a small part of the set is considered at any one time. This is what gives it its 'local quality' and what leads to many of its interesting features; see Figure 2.1.

The *Assouad dimension* of a non-empty set $F \subseteq \mathbb{R}^d$ is defined by

$$\dim_{\mathrm{A}} F = \inf \Big\{ \alpha : \text{ there exists a constant } C > 0 \text{ such that, for all } 0 < r < R \text{ and } x \in F, \; N_r\big(B(x,R) \cap F\big) \leqslant C \left(\frac{R}{r}\right)^{\alpha} \Big\}.$$

Recall that $N_r(E)$ is the smallest number of open sets required for an r-cover of a bounded set E. Note that we can replace N_r in the definition of the Assouad dimension with any of the standard covering or packing functions, see the discussion in Chapter 1, and still obtain the same value for the dimension. For example, $N_r(E)$ could denote the number of cubes in an r-mesh oriented at the origin which intersect E or the maximum cardinality of an r-separated subset of E. We also obtain an equivalent definition if the ball $B(x, R)$ is taken to be open or closed (although we usually think of it as being closed) or if it is replaced by a cube of sidelength R centred at x.

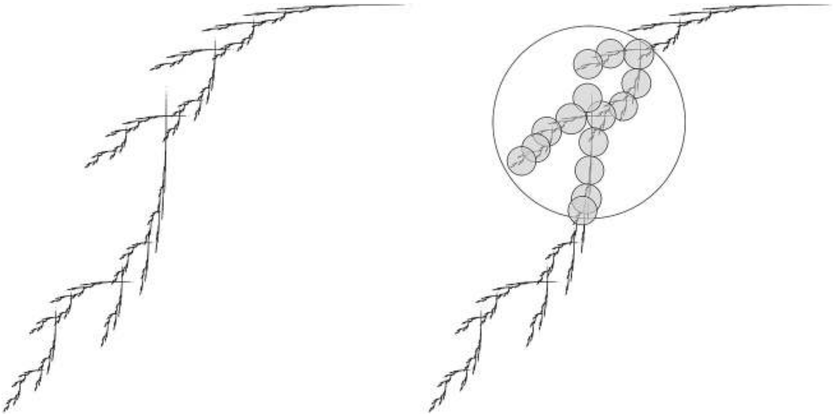

Figure 2.1 Left: the self-affine set from Figure 1.2. Right: A covering of a ball intersected with a 'thick part' of the self-affine set from Figure 1.2 using balls of smaller radii. Understanding how large such covers have to be leads to calculation of the Assouad dimension.

Before we move on let us consider a simple but fundamental example; see Figure 2.2. This example serves to demonstrate how inhomogeneity across a set can cause the box and Assouad (and Hausdorff) dimensions to be distinct.

Theorem 2.1.1 *For* $F = \{0\} \cup \{1/n : n \in \mathbb{N}\}$,

$$\dim_{\mathrm{A}} F = 1,$$

$$\dim_{\mathrm{B}} F = 1/2,$$

and

$$\dim_{\mathrm{H}} F = 0.$$

Proof To prove that $\dim_{\mathrm{A}} F = 1$ it suffices to find a constant $c > 0$ and a sequence of points $x_n \in F$ and scales $0 < r_n < R_n$ such that $R_n/r_n \to \infty$ and for all n

$$N_{r_n}(B(x_n, R_n) \cap F) \geqslant cR_n/r_n.$$